# Some aspects of computational mechanics: From elasticity to plasticity

Student project

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# Elasticity: Modeling

Lamé equation:

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\begin{aligned}
(\lambda+\mu)(\nabla \operatorname{div} u)^T + \mu \triangle u &= -f \quad \text{in } \Omega, && (1)\\
\varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), && (2)\\
\sigma &= \lambda \operatorname{tr}(\varepsilon(u)) I + 2\mu\varepsilon(u), && (3)\\
\sigma \cdot n &= g \quad \text{on } \Gamma_N, && (4)\\
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Weak formulation: Find $u \in H^1(\Omega)$

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\begin{gathered}
\int_\Omega \varepsilon(u) : \mathbb{C}\varepsilon(v)\, \mathsf{d}x = \int_\Omega f \cdot v \,\mathsf{d}x + \int_{\Gamma_N} g \cdot v \,\mathsf{d}x \\
\forall v \in H^1_D(\Omega) := \{v \in H^1(\Omega) : Mv = 0 \text{ on } \Gamma_D\}
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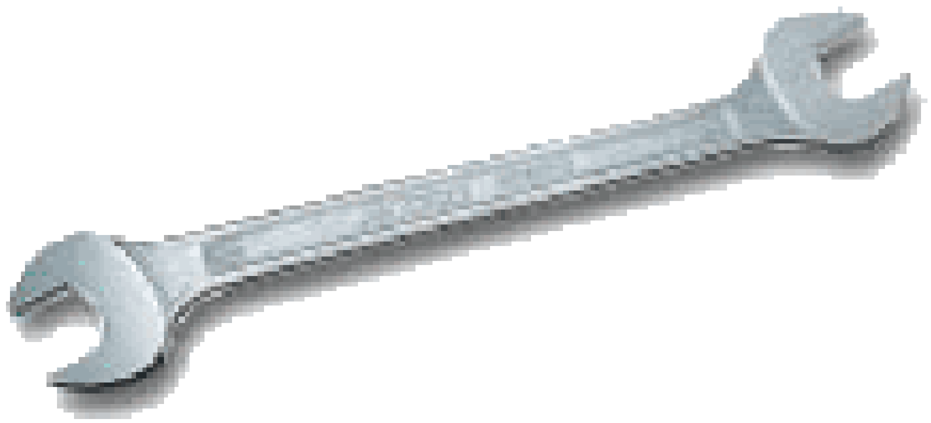
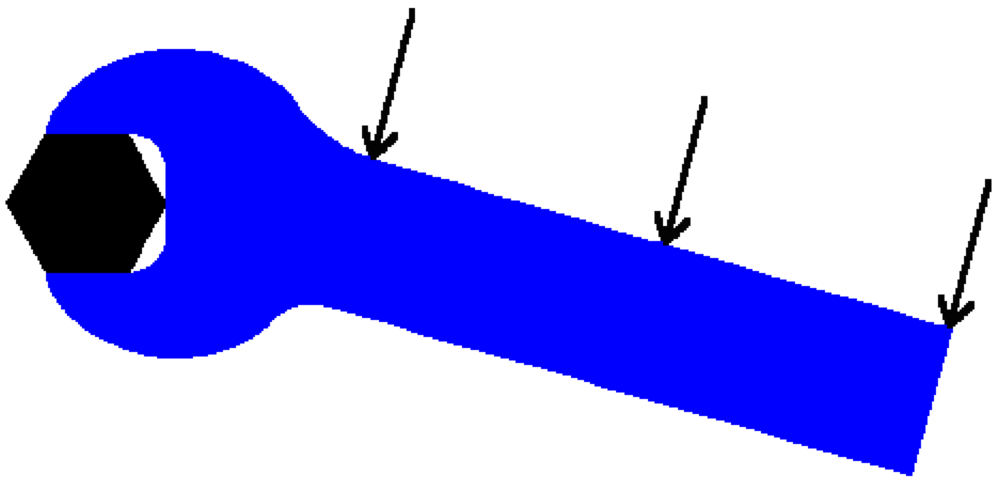
Exercise: $\sigma := 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)I, \varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T) : \quad \operatorname{div}\sigma = ?$

# Screw wrench



## Boundary value problem + mesh



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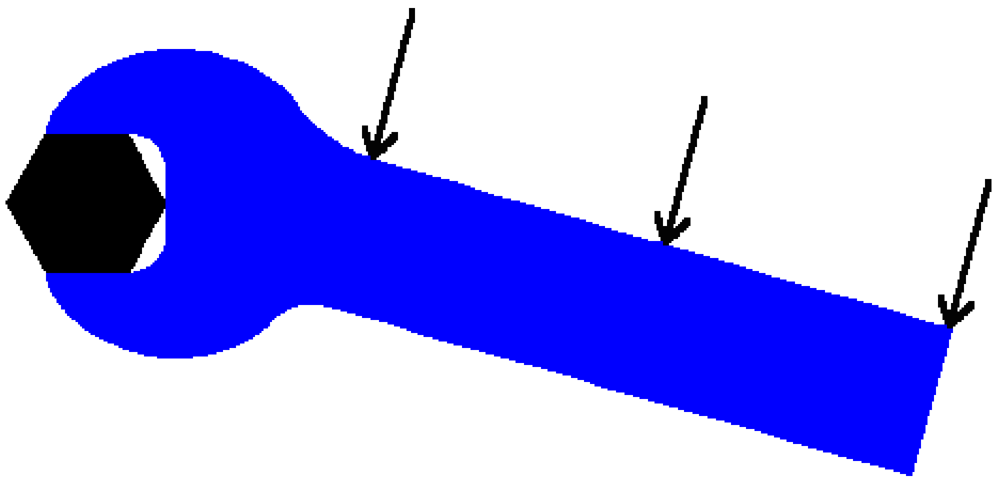

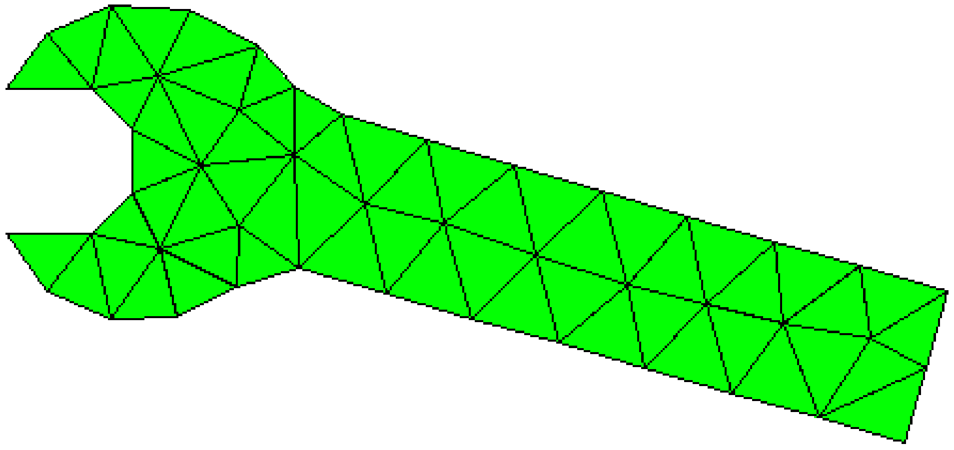

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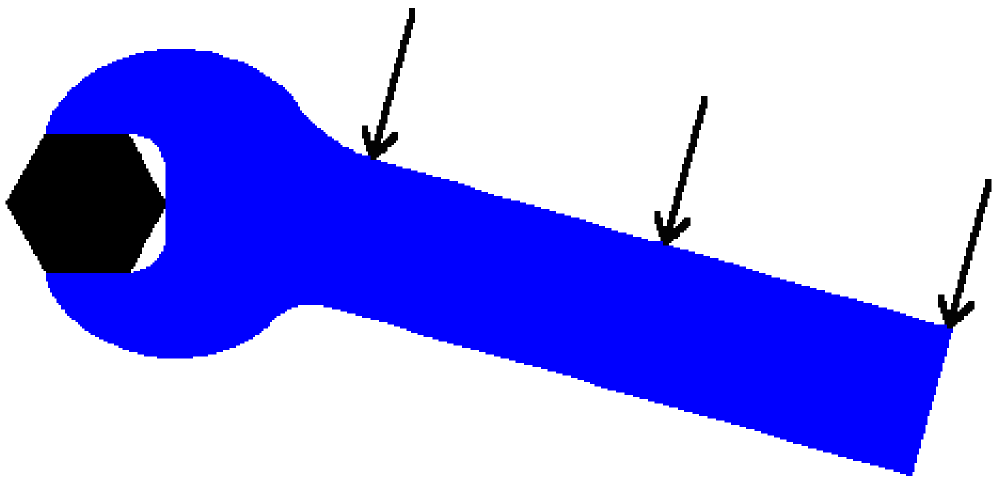
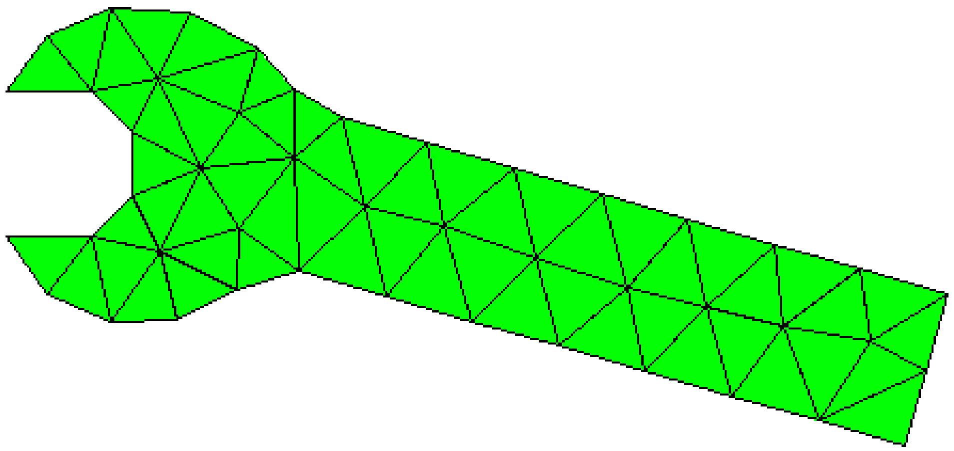
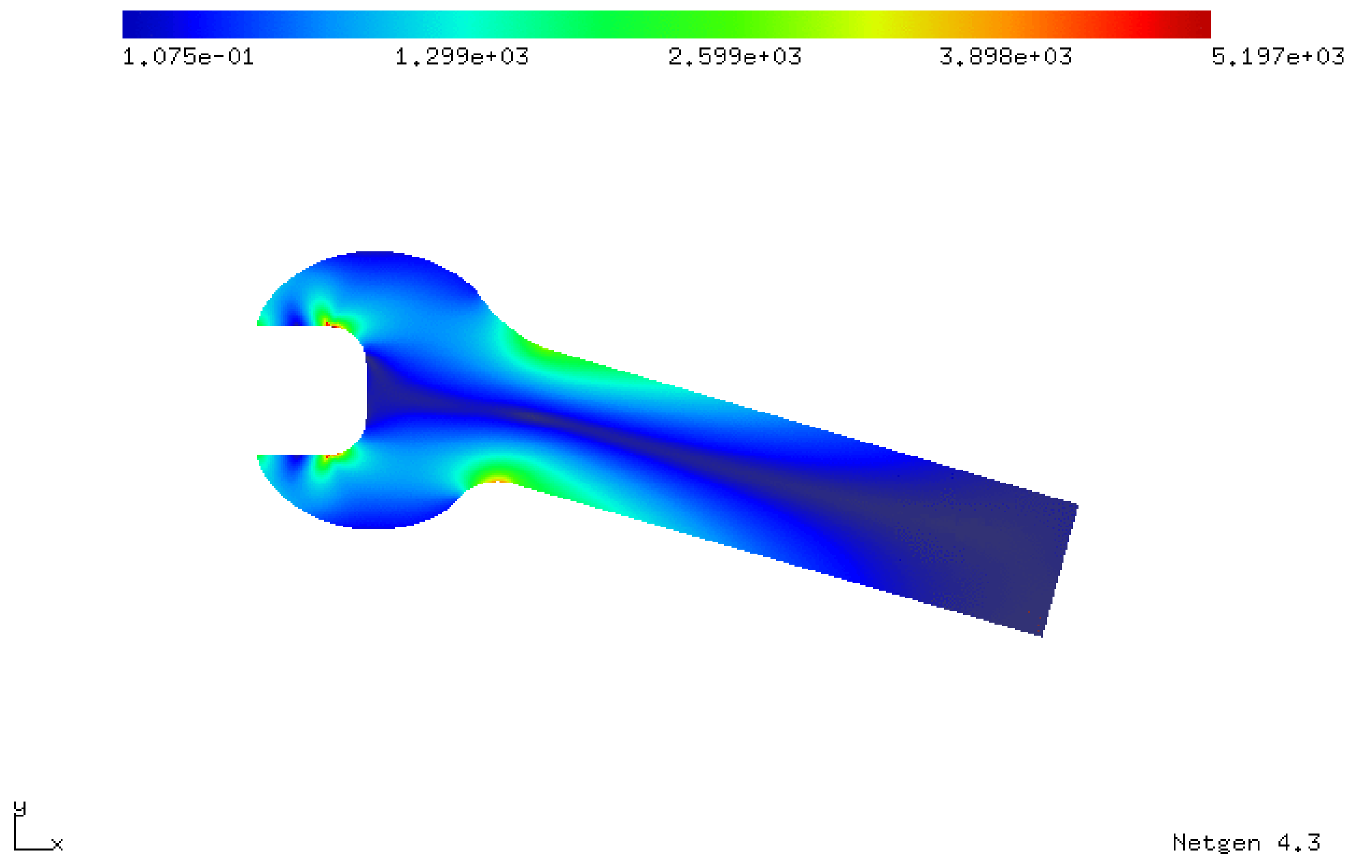




- Regular triangulation (generated by NETGEN)

## Finite Element solution: Equivalent stress



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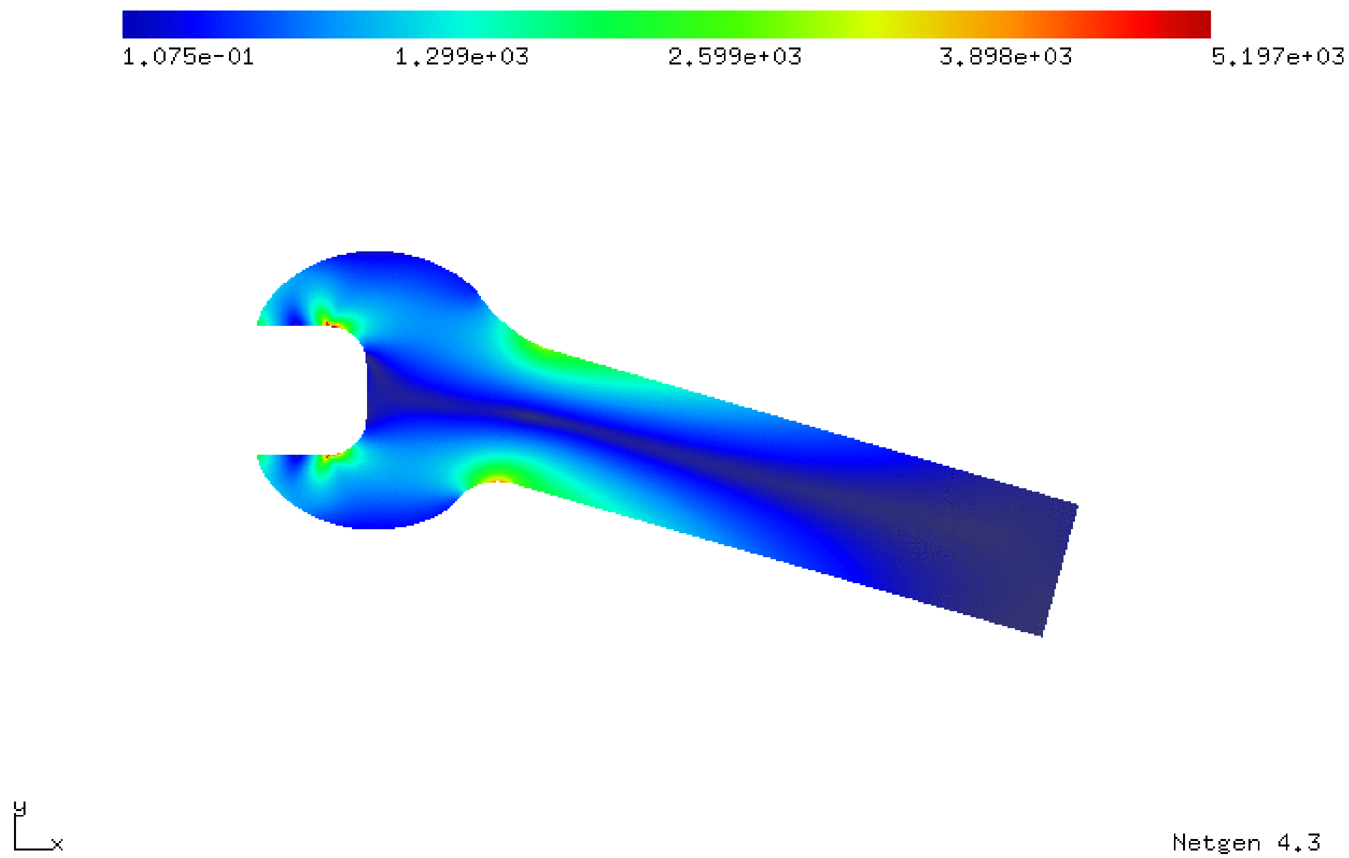


- High stresses yield plastic deformations : need of elastoplastic model!

# Elastoplasticity: Modeling

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\begin{aligned}
\varepsilon(u) &= \mathbb{C}^{-1}\sigma + p, & (12)\\
\varphi(\sigma, \alpha) &< \infty, & (13)\\
\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha) & (14)
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# Some convex analysis

**Definition 1 (indicator function, conjugate function).** *Let $Y \subset X$ be a convex set, $x \in Y$. For any set $S \subset X$, the* indicator function *$I_S$ of $S$ is defined by*

$$I_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S. \end{cases} \tag{15}$$

*For a function $f : X \to [-\infty, \infty]$ we define the* conjugate function *$f^* : X^* \to [-\infty, \infty]$ by*

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle - f(x)). \tag{16}$$

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**Definition 2 (subdifferential).** *Let $f$ be a convex function on $X$. For any $x \in X$ the* subdifferential *$\partial f(x)$ of $x$ is the possibly empty subset of $X^*$ defined by*

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Exercise: What is $\partial|x|$?

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Exercise: Show that $\frac{x}{||x||} \in \partial || \cdot ||(x)$.

# Convex analysis: dual formulation

**Definition 3 (lower semicontinuity).** *A function* $f : X \to [-\infty, +\infty]$ *is called* lower semicontinuous *if*

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Application to plasticity:

$$\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) \leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha) \quad \forall \tau, \beta$$

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# Isotropic hardening

Yield function $\Phi(\sigma, \alpha)$ for $\alpha \in \mathbb{R}_0^+$ and material parameters $H, \sigma_y \in \mathbb{R}_0^+$

$$\Phi(\sigma, \alpha) = |\operatorname{dev} A| - \sigma_y(1 + H\alpha)$$

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Dissipation functional

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Exercise:

$$\varphi^*(A, B) = \begin{cases} \sigma_y|A| & \text{if } \operatorname{tr} A = 0, \text{ and } B + H\sigma_y|A| \leq 0, \\ +\infty & \text{if } \operatorname{tr} A \neq 0 \text{ and } B + H\sigma_y|A| > 0. \end{cases}$$

# One time step problem

Find $(u, p)$ such that

$$\begin{aligned} f(u,p) :=& \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(u) - p)\,\mathsf{d}x + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p - p_0|)^2\,\mathsf{d}x + \int_\Omega \sigma_y |p - p_0|\,\mathsf{d}x \\ & - \int_\Omega f u\,\mathsf{d}x \longrightarrow \min \end{aligned}$$

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Minimization in $\tilde{p}$: For each integration point

$$F(\tilde{p}) := \frac{1}{2}(\mathbb{C} + \sigma_y^2 H^2 \mathbb{I})\tilde{p} : \tilde{p} - \mathbb{C}[\varepsilon(u) - p_0] : \tilde{p} + \sigma_y(1+\alpha_0 H)|\tilde{p}| \longrightarrow \min,$$

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Unique solution:

$$\tilde{p} = \frac{(||\,\mathrm{dev}\,A|| - b)_+}{2\mu + \sigma_y^2 H^2} \frac{\mathrm{dev}\,A}{||\,\mathrm{dev}\,A||}$$

where

$$A = \mathbb{C}[\varepsilon(u) - p_0], b = \sigma_y(1 + \alpha_0 H).$$

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Matrix notation: $\varepsilon(u) = Bu$, $|\tilde{p}| = (\tilde{p}^T Q \tilde{p})^{1/2}$

Exercise: Derive $Q$ and $B$.

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Regularization: $|p| \to |p|_\epsilon$: absolute (norm) value regulator is defined for $a \in \mathbb{R}_0^+, \epsilon \geq 0$ as

$$a_\epsilon = \begin{cases} a & \text{if } a \geq \epsilon, \\ \frac{1}{2\varepsilon}a^2 + \frac{\epsilon}{2} & \text{if } a < \epsilon. \end{cases}$$

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A necessary condition of the minima is $f'(u,\tilde{p}) = 0$, i.e.,

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -f - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix} = 0.$$

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Matrix notation: $\varepsilon(u) = Bu$, $|\tilde{p}| = (\tilde{p}^T Q \tilde{p})^{1/2}$
Exercise: Derive $Q$ and $B$.

$$f(u, \tilde{p}) = \frac{1}{2} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix}^T \begin{pmatrix} B^T \mathbb{C} B & -B^T \mathbb{C} \\ -\mathbb{C} B & \mathbb{C} + \mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -f - B^T \mathbb{C} p_0 \\ \mathbb{C} p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \frac{1}{2} \mathbb{C} \tilde{p}_0 : \tilde{p}_0 + \frac{1}{2} \alpha_0^2 \rightarrow \text{min},$$

where $\mathbb{H} = \sigma_y^2 H^2 Q + 2\sigma_y (1 + \alpha_0 H) \frac{Q}{|p|}$.

Regularization: $|p| \longrightarrow |p|_\epsilon$: absolute (norm) value regulator is defined for $a \in \mathbb{R}_0^+, \epsilon \geq 0$ as

$$a_\epsilon = \begin{cases} a & \text{if } a \geq \epsilon, \\ \frac{1}{2\varepsilon} a^2 + \frac{\epsilon}{2} & \text{if } a < \epsilon. \end{cases}$$

A necessary condition of the minima is $f'(u, \tilde{p}) = 0$, i.e.,

$$\begin{pmatrix} B^T \mathbb{C} B & -B^T \mathbb{C} \\ -\mathbb{C} B & \mathbb{C} + \mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -f - B^T \mathbb{C} p_0 \\ \mathbb{C} p_0 \end{pmatrix} = 0.$$

Schur Complement system:

$$B^T (\mathbb{C} - \mathbb{C}(\mathbb{C} + \mathbb{H})^{-1}) B u = f + B^T (\mathbb{C} - \mathbb{C}(\mathbb{C} + \mathbb{H})^{-1} \mathbb{C}) p_0$$

$\mathrm{tr}\,\tilde{p} = 0 \longrightarrow$ substitution $\tilde{p} = P\bar{p}$ with

$$\text{2D: } P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{3D: } P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\operatorname{tr}\tilde{p} = 0 \longrightarrow$ substitution $\tilde{p} = P\bar{p}$ with

$$\text{2D: } P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{3D: } P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$B^T(C - CP(P^T(C+H)P)^{-1}P^TC)Bu = f + B^T(C - CP(P^T(C+H)P)^{-1}P^TC)\,p_0.$$