Symbolic techniques applied to the calculation of two-yield elastoplastic dependance

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F1306 - Adaptive Multilevel Methods for
Nonlinear 3D Mechanical Problems

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F1302 - THEOREMA: Proving, Solving, and Computing
in the Theory of Hilbert Spaces

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Outline

- Reduction of the elastoplastic system to the system of polynomials (F1306)
- Numerical iterative method (F1306)
- Gröbner basis solution (F1302)

Continuous model of two-yield elastoplasticity

Problem (PI): For $l \in H^1(0,T;\mathcal{H}^*), l(0) = 0$, find $w = (u, p_1, p_2) : [0,T] \to \mathcal{H}, w(0) = 0$ s. t.

$$\langle l(t), z - \dot{w}(t) \rangle \leq a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t))$$

$\forall z = (v, \tau_1, \tau_2) \in \mathcal{H}$.

$$\begin{aligned}
\mathcal{H} &= H^1_D(\Omega) \times L^2(\Omega)^{d\times d}_{sym} \times L^2(\Omega)^{d\times d}_{sym}, \\
a(w,z) &= \int_\Omega \Big(\mathbb{C}(\varepsilon(u) - \sum_{i=1}^2 p_i)\Big) : \Big(\varepsilon(v) - \sum_{i=1}^2 \tau_i\Big)\,\mathrm{d}x + \sum_{i=1}^2 \int_\Omega \mathbb{H}_i p_i : \tau_i\,\mathrm{d}x, \\
\langle l(t), z \rangle &= \int_\Omega f(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}x, \\
j(z) &= \int_\Omega \sum_{i=1}^2 D_i(\tau_i)\,\mathrm{d}x, \quad D_i(x) = \begin{cases} \sigma_y^i ||x|| & \text{if } \operatorname{tr} x = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad \text{(von Mises)}
\end{aligned}$$

Discrete model of two-yield elastoplasticity

Problem ($PI_{discrete}$): Given $P_1^0, P_2^0 \in \text{dev } \mathcal{S}^0(\mathcal{T})^{d\times d}_{sym}$, find $U^1 \in \mathcal{S}^1_D(\mathcal{T})$ s. t.

$$\int_\Omega \mathbb{C}(\epsilon(U^1) - P_1^1 - P_2^1) : \epsilon(V)\, \mathrm{d}x - \int_\Omega f(t) V\, \mathrm{d}x - \int_{\Gamma_N} gV\, \mathrm{d}x = 0 \quad \forall V \in S^1_D(\mathcal{T}),$$

where $P = (P_1, P_2)^T = (P_1^1, P_2^1)^T - (P_1^0, P_2^0)^T$ minimizes on every element $T \in \mathcal{T}$

$$\min_Q \frac{1}{2}(\hat{\mathbb{C}} + \hat{\mathbb{H}})Q : Q - A|_T : Q + ||Q||_{\sigma_y},$$

$$\forall Q = (Q_1, Q_2)^T, Q_1, Q_2 \in \mathbb{R}^{d\times d}_{sym}, \mathrm{tr} Q_1 = \mathrm{tr} Q_2 = 0.$$

$$A = \begin{pmatrix} \mathbb{C}\epsilon(U^1) \\ \mathbb{C}\epsilon(U^1) \end{pmatrix} - (\hat{\mathbb{C}} + \hat{\mathbb{H}}) \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix}, \quad \hat{\mathbb{C}} + \hat{\mathbb{H}} = \begin{pmatrix} \mathbb{C} + \mathbb{H}_1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}_2 \end{pmatrix}, ||(Q_1, Q_2)||_{\sigma_y} = \sigma_y^1 ||Q_1|| + \sigma_y^2 ||Q_2||.$$

Analytical approach for minimization of $f(Q) = \frac{1}{2}(\hat{\mathbb{C}} + \hat{\mathbb{H}})Q : Q - Q : A + ||Q||_{\sigma_y}$

f has a subdifferential, i.e.,

$$\partial f(P) = (\hat{\mathbb{C}} + \hat{\mathbb{H}})P - A + \partial || \cdot ||_{\sigma_y}(P)$$

Mininum condition on P

$$0 \in \partial f(P) \Leftrightarrow A - (\hat{\mathbb{C}} + \hat{\mathbb{H}})P \in \partial || \cdot ||_{\sigma_y}(P)$$

In case $P_1 \neq 0, P_2 \neq 0$ is

$$\partial || \cdot ||_{\sigma_y}(P) = \{\sigma_y^1 \frac{P_1}{||P_1||}, \sigma_y^2 \frac{P_2}{||P_2||}\}$$

Nonlinear system in P_1, P_2

$$\begin{pmatrix} \mathrm{dev}\, A_1 \\ \mathrm{dev}\, A_2 \end{pmatrix} - \begin{pmatrix} (2\mu + h_1)\mathbb{I} & 2\mu\mathbb{I} \\ 2\mu\mathbb{I} & (2\mu + h_2)\mathbb{I} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \sigma_y^1 \frac{P_1}{||P_1||} \\ \sigma_y^2 \frac{P_2}{||P_2||} \end{pmatrix}$$

Substitutions $P_1 = \xi_1 X_1, P_2 = \xi_2 X_2$

$$\begin{pmatrix} \mathrm{dev}\, A_1 \\ \mathrm{dev}\, A_2 \end{pmatrix} = \begin{pmatrix} (\sigma_y^1 + (2\mu + h_1)\xi_1)\mathbb{I} & 2\mu\xi_2\mathbb{I} \\ 2\mu\xi_1\mathbb{I} & (\sigma_y^2 + (2\mu + h_2)\xi_2)\mathbb{I} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Additional substitutions $\eta_1 := \sigma_y^1 + (2\mu + h_1)\xi_1, \eta_2 := \sigma_y^2 + (2\mu + h_2)\xi_2, \nu_1 := 2\mu\xi_1, \nu_2 := 2\mu\xi_2$ yield

$$\eta_2 \operatorname{dev} A_1 - \nu_2 \operatorname{dev} A_2 = (\eta_1\eta_2 - \nu_1\nu_2)X_1, \tag{1}$$

$$-\nu_1 \operatorname{dev} A_1 + \eta_1 \operatorname{dev} A_2 = (\eta_1\eta_2 - \nu_1\nu_2)X_2. \tag{2}$$

Normalization of (1) and (2) and the application of substitutions for $\eta_1, \eta_2, \nu_1, \nu_2$ give the system of nonlinear equations for positive ξ_1, ξ_2

$$|l_1(\xi_1)| - |r(\xi_1, \xi_2)| = 0, \quad |l_2(\xi_2)| - |r(\xi_1, \xi_2)| = 0, \tag{3}$$

where

$$l_1(\xi_1) = (\sigma_y^1 + (2\mu + h_1)\xi_1) \operatorname{dev} A_2 - 2\mu\xi_1 \operatorname{dev} A_1,$$
$$l_2(\xi_2) = (\sigma_y^2 + (2\mu + h_2)\xi_2) \operatorname{dev} A_1 - 2\mu\xi_2 \operatorname{dev} A_2,$$
$$r(\xi_1, \xi_2) = (\sigma_y^1 + (2\mu + h_2)\xi_1)(\sigma_y^2 + (2\mu + h_2)\xi_2) - 4\mu^2\xi_1\xi_2.$$

Instead of the solving (3) we prefer to solve the equivalent system of nonlinear equations

$$\Phi_1(\xi_1, \xi_2) = |l_1(\xi_1)|^2 - (r(\xi_1, \xi_2))^2 = 0, \Phi_2(\xi_1, \xi_2) = |l_2(\xi_1)|^2 - (r(\xi_1, \xi_2))^2 = 0. \tag{4}$$

The first polynomial system of two variables in ξ_1 and ξ_2

Substitutions

$$A = |\sigma_y^1 \,\mathrm{dev}\, A_2|^2, D = |\sigma_y^2 \,\mathrm{dev}\, A_1|^2$$
$$B = 2\sigma_y^1 \,\mathrm{dev}\, A_2 : ((2\mu + h_1) \,\mathrm{dev}\, A_2 - 2\mu \,\mathrm{dev}\, A_1), E = 2\sigma_y^2 \,\mathrm{dev}\, A_1 : ((2\mu + h_2) \,\mathrm{dev}\, A_1 - 2\mu \,\mathrm{dev}\, A_2)$$
$$C = |(2\mu + h_1) \,\mathrm{dev}\, A_2 - 2\mu \,\mathrm{dev}\, A_1|^2, F = |(2\mu + h_2) \,\mathrm{dev}\, A_1 - 2\mu \,\mathrm{dev}\, A_2|^2$$
$$G = (\sigma_y^1 \sigma_y^2)^2 > 0$$
$$H = \sigma_y^2 (2\mu + h_1) > 0, I = \sigma_y^1 (2\mu + h_2) > 0$$
$$J = 2\mu(h_1 h_2) + h_1 h_2 > 0.$$

Φ_1, Φ_2 are polynomials of the second degree in two variables ξ_1, ξ_2

$$\Phi_1(\xi_1, \xi_2) = A + B\xi_1 + C\xi_1^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0 \tag{5}$$
$$\Phi_2(\xi_1, \xi_2) = D + E\xi_2 + F\xi_2^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0 \tag{6}$$

Example with know exact solution

Example 1. Let $\mu = 1, \sigma_y^1 = 1, \sigma_y^2 = 2, h_1 = 1, h_2 = 1$ and $A_1 = A_2 = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix}$. Nonlinear system of equations (4) for positive $\xi_1, \xi_2 > 0$ reads

$$\Phi_1(\xi_1, \xi_2) \quad = 200 + 400\,\xi_1 + 200\,\xi_1^2 - (2 + 6\,\xi_1 + 3\,\xi_2 + 5\,\xi_1\,\xi_2)^2 = 0, \tag{7}$$

$$\Phi_2(\xi_1, \xi_2) \quad = 800 + 800\,\xi_2 + 200\,\xi_2^2 - (2 + 6\,\xi_1 + 3\,\xi_2 + 5\,\xi_1\,\xi_2)^2 = 0. \tag{8}$$

ξ_1 is expressed from (8)

$$\xi_1 = \frac{-2 - 3\xi_2 \pm 10\sqrt{2}(2 + \xi_2)}{6 + 5\xi_2} \tag{9}$$

and the substitution of (9) into (7) implies after the factorization the equality

$$\frac{(5\xi_2 + 8 \pm 10\sqrt{2})(5\xi_2 + 4 \mp 10\sqrt{2})(\xi_2 + 2)^2}{(6 + 5\,\xi_2)^2} = 0. \tag{10}$$

the different signs of ξ_1 and ξ_2. The roots of (10) are given by

$$\xi_2 = \{-\frac{4}{5} + 2\sqrt{2}, -\frac{8}{5} - 2\sqrt{2}, -2, -2\} \cup \{-\frac{4}{5} - 2\sqrt{2}, -\frac{8}{5} + 2\sqrt{2}, -2, -2\}$$

There are two positive roots

$$\xi_2 = -\frac{4}{5} + 2\sqrt{2} \approx 2.028427124 \quad \text{and} \quad \xi_2 = -\frac{8}{5} + 2\sqrt{2} \approx 1.228427124$$

Substitution of $\xi_2 = -\frac{4}{5} + 2\sqrt{2}$ into (7) yields the quadratic equation

$$(995\xi_1 + 801 - 50\sqrt{2})(5\xi_1 - 1 - 10\sqrt{2}) = 0$$

with roots $\xi_1 = \{\frac{1}{5} + 2\sqrt{2}, -\frac{801}{995} + \frac{10}{199}\sqrt{2}\}$ where only the first root

$$\xi_1 = \frac{1}{5} + 2\sqrt{2} \approx 3.028427125$$

is positive. Note that substitution of $\xi_2 = -\frac{8}{5} + 2\sqrt{2}$ into (7) yields another quadratic equation with negative roots only. Therefore, there is one positive solution pair

$$(\xi_1, \xi_2) = (3.028427125, 2.028427124).$$

Iterative method

The (numerical) Algorithm (J. Valdman PhD thesis) with $tol = 10^{-12}$ and the initial approximation

$$P_2^0 = \frac{(|\operatorname{dev} A_2| - \sigma_y^2)_+}{2\mu + h_2} \frac{\operatorname{dev} A_2}{|\operatorname{dev} A_2|} \quad \text{and} \quad P_1^0 = \frac{(|\operatorname{dev} A_1 - 2\mu P_2^0| - \sigma_y^1)_+}{2\mu + h_1} \frac{\operatorname{dev} A_1 - 2\mu P_2^0}{|\operatorname{dev} A_1 - 2\mu P_2^0|}.$$

generates approximations $P_1^i, P_2^i, i = 1, 2, \ldots$ in the form

$$P_1^i = \begin{pmatrix} x^i & 0 \\ 0 & -x^i \end{pmatrix} \quad \text{and} \quad P_2^i = \begin{pmatrix} y^i & 0 \\ 0 & -y^i \end{pmatrix},$$

and terminates after 34 approximations (computational time 7.16×10^{-5} sec) with

$$P_1^{34} = \begin{pmatrix} 2.14142 & 0 \\ 0 & -2.14142 \end{pmatrix} \quad \text{and} \quad P_2^{34} = \begin{pmatrix} 1.43431 & 0 \\ 0 & -1.4343 \end{pmatrix}.$$

$$(||P_1^{34}||, ||P_2^{34}||) = (\xi_1, \xi_2) \approx (3.028425, 2.0284207)$$

SYNAPS package calculation (with project 1304) terminated after 0.05 sec with aproximately the same solution $(\xi_1, \xi_2) = (3.02843, 2.02843)$.