Spezialforschungsbereich F013

Subproject F1306

Adaptive Multilevel Methods for Nonlinear 3D Mechanical Problems

Computational plasticity

Mathematical model of elastoplasticity

Basic equations

The stress field of a deformed body in $\mathbb{R}^n$ has to satisfy

$$-\text{div } \sigma = b$$
$$\sigma = \sigma^T$$

with given body forces b. The linearized strain tensor is defined by

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

The phenomenon of plasticity is described by an additional non-linear term in the stress-strain relation

$$\varepsilon(u) = \mathbb{C}^{-1}\sigma + p$$

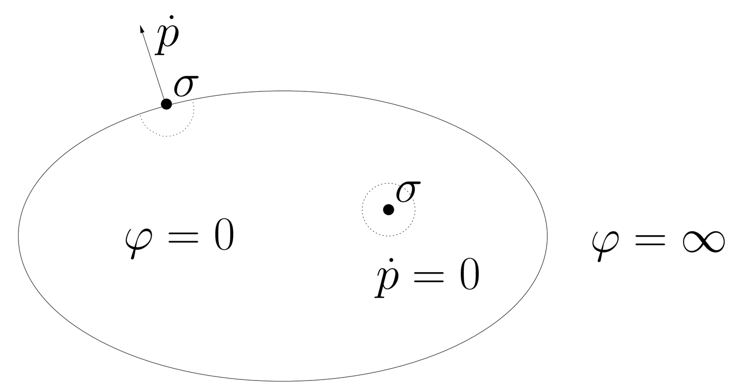
The admissible stresses are restricted by a yield function φ depending on the hardening of the material, the Prandtl-Reuß normality law describes the time development

$$\varphi(\sigma, \alpha) < \infty$$
$$\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) \leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha)$$

The hardening parameter α depends on the material law. $\dot{p}$ denotes the time derivative of the plastic strain p.

Normality law

If we consider the Prandtl-Reuß normality law without α, that is the case of perfect plasticity, then φ describes the domain where the stress is admissible:



Numeric-analytic steps

The time dependent variational inequality is solved by an implicit time discretization, e.g. an implicit Euler scheme. For given values at some time step t_0 the updated values for $t_1 = t_0 + \Delta t$ have to be determined.

The problem is reformulated by using functional-analytic arguments, i.e., the arguments in the variational inequality are switched using a dual functional. Then, an equivalent minimization problem can be derived: Find the minimizer (u, p, α) of

$$f(u,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx + \Delta t \int_\Omega \varphi^*\left(\frac{p - p_0}{\Delta t}, \frac{\alpha_0 - \alpha}{\Delta t}\right)dx - \int_\Omega b\, u\, dx$$

φ^* is the dual functional of φ.

Algorithm

Minimization problem for isotropic hardening

The dual functional can be computed and the minimization problem simplifies and writes as: Find the minimizer (u, p) of

$$f(u,p) := \tfrac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx - \int_\Omega b\, u\, dx + \tfrac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p - p_0|)^2 dx + \int_\Omega \sigma_y |p - p_0| dx$$

under the constraint tr $(p - p_0) = 0$. Define $\tilde{p} = p - p_0$. f is a convex, non-differentiable function with quadratic terms. It is regularized by smoothing the sharp bend of the absolute value:

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon \end{cases}$$

The minimization strategy in each time step is

$$u^{k+1} = \text{argmin}_v \min_q \tilde{f}(v,q) = \text{argmin}_v \tilde{f}(v, q_{\text{opt}}(v))$$

Minimization in u

The Finite-Element-Method discretization of the unconstrained objective in matrix form is equivalent to

$$\frac{1}{2}\begin{pmatrix} u \\ \tilde{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C} + \mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} \longrightarrow \min!$$

$\mathbb{H}$ is the Hessian with respect to p. Necessary condition:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C} + \mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix} = 0$$

The Schur-Complement system in u with the matrix

$$S = B^T(\mathbb{C} - \mathbb{C}(\mathbb{C} + \mathbb{H})^{-1}\mathbb{C})B$$

is solved by multigrid preconditioned conjugate gradient method.

Minimization problem in p

Minimizing p with the Schur-Complement system would be an inexact and slow procedure. The minimization can be done locally in each integration point using Newton's method.

Constraint

Since the constraint tr $\tilde{p}$ is linear

$$\text{2D: } \tilde{p}_{22} = -\tilde{p}_{11}$$
$$\text{3D: } \tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22}$$

the minimization problems can be projected onto a hyperplane, where the constraint is satisfied exactly: e.g.

$$S = B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C} + \mathbb{H})P)^{-1}P^T\mathbb{C})B$$

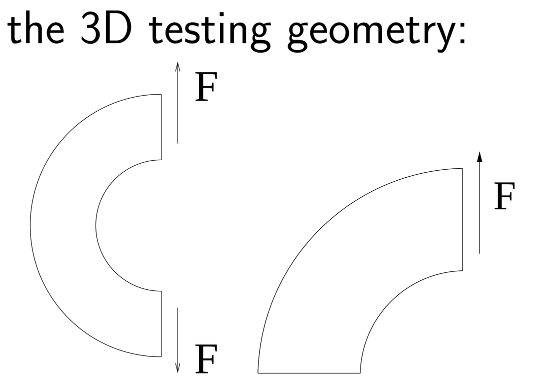
with the projection matrix P.

Results and future work

- NGSolve - finite element package
- FEM basis functions: piecewise quadratic
- Full multigrid method

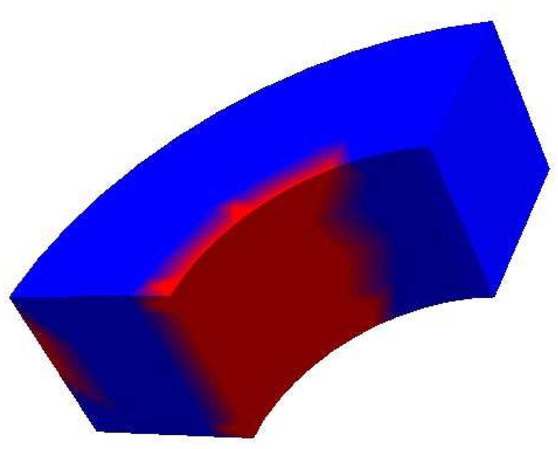
Testing geometry

2D sketches of the 3D testing geometry:



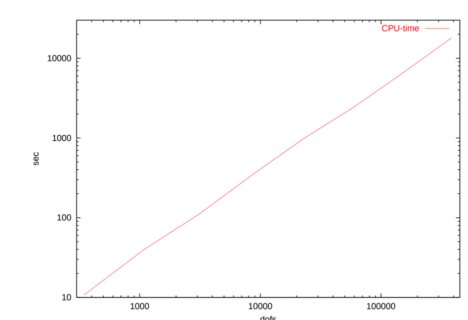
Plasticity domain

The material in the red domain is permanently deformed.

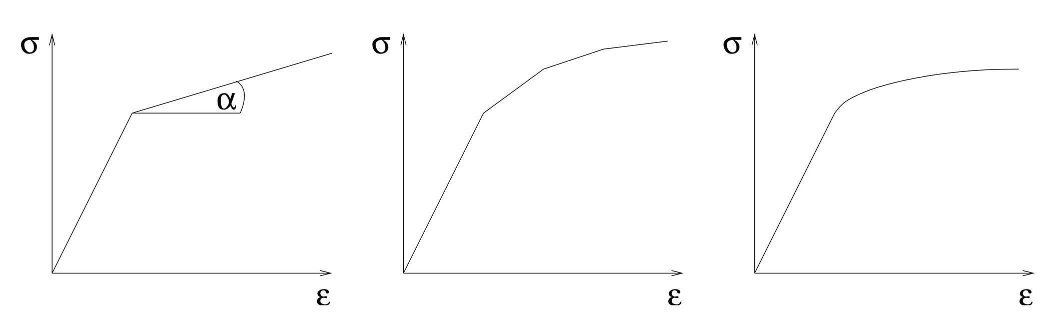


Complexity

The CPU-time depends linearly on the number of unknowns dofs.



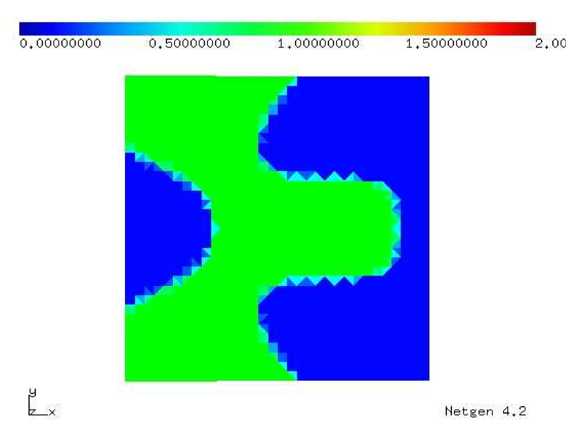
Multi-yield (Two-yield) plasticity



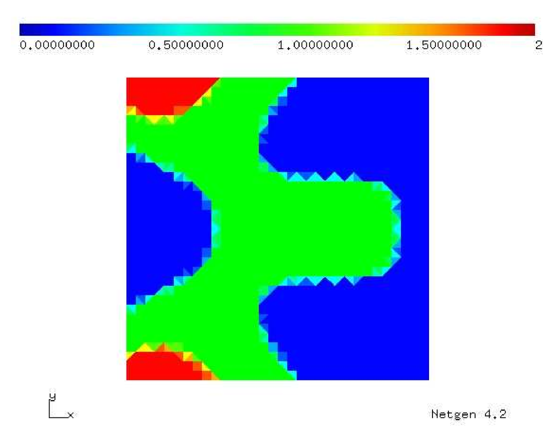
Single yield　　Multi yield　　Reality

Elastoplastic domains

blue elastic, green first plastic, red second plastic



Kinematic hardening



Two-yield hardening

Outlook

- Convergence proof of the algorithm
- Extension to other hardening laws
- Exact analytic formulas for minimizing p
- Nonlinear hardening, big deformations

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