COMMUTING QUASI-INTERPOLATION OPERATORS FOR MIXED FINITE ELEMENTS

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ABSTRACT. Conforming finite elements for $H(curl)$ and $H(div)$ spaces became a main research topics in numerical analysis. The so called de Rham diagram [5, 8, 7, 4] relates the exact sequence of continuous spaces $H^1 \to H(curl) \to H(div) \to L^2$ to their corresponding discrete counterparts. Up to now, only the local nodal interpolation operators, and global Fortin operators [3] have been known to fulfill the commuting diagram property. In this paper, new quasi-local, Clément-type operators satisfying the commuting diagram property are introduced. The result, in particular, should help to generalize and simplify existing multigrid theories as well as a posteriori error estimates for Maxwell's equations.

1. INTRODUCTION

The de Rham diagram compactly visualizes this paper's results:

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\big\downarrow \Pi^W & & \big\downarrow \Pi^Q & & \big\downarrow \Pi^V & & \big\downarrow \Pi^W \\
W_h & \xrightarrow{\nabla} & Q_h & \xrightarrow{\text{curl}} & V_h & \xrightarrow{\text{div}} & S_h\,.
\end{array} \tag{1}
$$

The function spaces of the first row,

$$
\begin{aligned}
W := H^1 \quad &:= \quad \{w \in L_2(\Omega) : \nabla w \in [L_2]^3\}, \\
Q := H(\text{curl}) \quad &:= \quad \{q \in [L_2(\Omega)]^3 : \text{curl}\, q \in [L_2]^3\}, \\
V := H(\text{div}) \quad &:= \quad \{v \in [L_2(\Omega)]^3 : \text{div}\, v \in L_2\}, \\
S := L_2(\Omega) &
\end{aligned}
$$

form a sequence on the continuous level. The spaces in the second row are the canonical finite element spaces. Here, we restrict ourselves

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to the lowest order elements on tetrahedral meshes, namely piecewise linears for W_h, lowest order Nédélec elements of first type [10] for Q_h, lowest order Raviart-Thomas elements [11] for V_h, and piecewise constants to generate S_h.

This paper presents new interpolation operators relating the continuous spaces to the discrete counterparts. Our operators are well defined on the Lebesgue spaces, and each consecutive pair fulfills the commuting diagram property,

$$\nabla \Pi^W = \Pi^Q \nabla, \quad \operatorname{curl} \Pi^Q = \Pi^V \operatorname{curl}, \quad \text{and} \quad \operatorname{div} \Pi^V = \Pi^S \operatorname{div}. \tag{2}$$

Additionally, local approximation properties are proven. Best to our knowledge, only the local nodal interpolation operators, which are defined on more regular sub-spaces [5], and the global Fortin operators [3] were known to fulfill the commuting diagram property.

These new operators are, beyond possibly many other applications, useful to simplify and generalize existing multigrid theory [9, 1] and a posteriori error estimates [2].

2. Definitions

The domain Ω is assumed to be polyhedral and with Lipschitz boundary. It is covered by a shape regular tetrahedral mesh. We define

$$\begin{aligned} \text{the set of vertices} \quad & \mathcal{V} = \{V_i\}, \\ \text{the set of edges} \quad & \mathcal{E} = \{E_{ij}\}, \\ \text{the set of faces} \quad & \mathcal{F} = \{F_{ijk}\}, \\ \text{the set of tetrahedra} \quad & \mathcal{T} = \{T_{ijkl}\}. \end{aligned}$$

The two indices of an oriented edge specify the initial vertex and final vertex. The three (four) indices of a face (a tetrahedron) specify the vertices and the orientation. Using the notation $[\cdot]$ for the convex hull we can write

$$E_{ij} = [V_i, V_j], \quad F_{ijk} = [V_i, V_j, V_k], \quad \text{and} \quad T_{ijkl} = [V_i, V_j, V_k, V_l].$$

The tangential unit vector τ of the edge E_{ij} is a positive multiple of the vector $V_j - V_i$, and the normal unit vector ν of the face F_{ijk} is a positive multiple of the vector $(V_j - V_i) \times (V_k - V_i)$. We define the local mesh size h associated with an edge, a face or an element as its diameter, and the mesh size associated with a vertex as the maximal mesh size of adjacent elements. Let φ_i^W, φ_{ij}^Q, φ_{ijk}^V, and φ_{ijkl}^S be the nodal basis functions for the spaces W_h, Q_h, V_h, and S_h, respectively. They are associated with a vertex, an edge, a face, and an element, respectively.

Then, the nodal interpolation operators are

$$\begin{aligned}
(I^W w)(x) &:= \sum_{V_i \in \mathcal{V}} w(V_i)\varphi_i^W(x), \\
(I^Q q)(x) &:= \sum_{E_{ij} \in \mathcal{E}} \int_{E_{ij}} \tau \cdot q \, ds \; \varphi_{ij}^Q(x), \\
(I^V v)(x) &:= \sum_{F_{ijk} \in \mathcal{F}} \int_{F_{ijk}} \nu \cdot v \, ds \; \varphi_{ijk}^V(x), \\
(I^S s)(x) &:= \sum_{T_{ijkl} \in \mathcal{T}} \int_{T_{ijkl}} s \, dx \; \varphi_{ijkl}^S(x).
\end{aligned}$$

It is well known and easily seen that these interpolation operators commute in the sense of (2). To overcome point evaluation which requires more regularity, local averaging operators [6, 12] for Sobolev spaces have been introduced. For each vertex V_i, let $\omega_i \subset \Omega \cup B(V_i, ch_i)$ be a set of non-zero measure. Next, fix some integer $p \geq 0$, and define for each V_i a function $f_i \in L_\infty(\omega_i)$ such that

$$\int_{\omega_i} f_i w \, dx = w(V_i) \qquad \text{for all polynomials } w \text{ up to order } p. \tag{3}$$

There is no need to specify ω_i and f_i exactly. One possibility is to choose balls ω_i, and fix f_i as unique polynomial of order p satisfying property (3). We assume

$$\|f_i\|_{L_\infty} \simeq h_i^{-3}, \tag{4}$$

which is usually proven by scaling arguments. There follows immediately $\|f_i\|_{L_1} \simeq 1$ and $\|f_i\|_{L_2} \simeq h_i^{-3/2}$. Note that we did not assume $V_i \in \omega_i$. The Clément type interpolation operator, defined on L_2, is

$$(\Pi^W w)(x) := \sum_{V_i \in \mathcal{V}} \Psi_i^W(w) \; \varphi_i^W(x) \qquad \text{with} \tag{5}$$

$$\Psi_i^W(w) := \int_{\omega_i} f_i(y) w(y) \, dy.$$

The new interpolation operators for the remaining spaces are all derived from the specific choice of the interpolation operator Π^W. Like the interpolation point for Π^W is smeared out, we now move all the involved vertices of the other operators:

$$(\Pi^Q q)(x) := \sum_{E_{ij} \in \mathcal{E}} \Psi_{ij}^Q(q) \; \varphi_{ij}^Q(x) \qquad \text{with} \tag{6}$$

$$\Psi^Q_{ij}(q) := \int\limits_{\omega_i}\int\limits_{\omega_j} f_i(y_1) f_j(y_2) \int\limits_{[y_1,y_2]} \tau \cdot q \, ds \; dy_2 dy_1.$$

We take all line integrals starting in the domain ω_i and terminating in ω_j, and average them by the weight functions f_i and f_j. Similarly, we define operators for $H(\text{div})$,

$$(\Pi^V v)(x) := \sum_{F_{ijk}\in\mathcal{F}} \Psi^V_{ijk}(v)\; \varphi^V_{ijk}(x) \qquad \text{with} \tag{7}$$

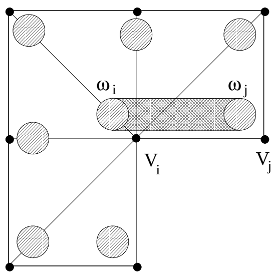
$$\Psi^V_{ijk}(v) = \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k} f_i(y_1) f_j(y_2) f_k(y_3) \int\limits_{[y_1,y_2,y_3]} \nu \cdot v \, ds \; dy_3 dy_2 dy_1,$$

and L_2,

$$(\Pi^S s)(x) = \sum_{T_{ijkl}\in\mathcal{T}} \Psi_{ijkl}(s)\; \varphi^S_{ijkl}(x) \qquad \text{with} \tag{8}$$

$$\Psi^S_{ijkl}(s) = \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k}\int\limits_{\omega_l} f_i(y_1) f_j(y_2) f_k(y_3) f_l(y_4) \int\limits_{\substack{[y_1,y_2,\\ y_3,y_4]}} s \, dx \; dy_4 dy_3 dy_2 dy_1.$$

We do not want to bother to much about boundary conditions. We assume that we have chosen the sets ω_i such that the involved integrals are all inside Ω, and thus the integrands have a legal domain. The sketch below shows the influence domain of the edge functional Ψ^Q_{ij}.



3. Properties of the interpolation operators

We start proving the algebraic property of the interpolation operators.

Theorem 1 (Commuting diagram property)**.** *The operators defined in Section 2 fulfill the commuting diagram property (2).*

Proof. We chose $w \in H^1$ and verify $\nabla \Pi^W w = \Pi^Q \nabla w$. Since both quantities are in Q_h, it is enough to check the unisolvent edge functionals

$$\int_{E_{ij}} \tau \cdot \nabla \Pi^W w \, ds = \int_{E_{ij}} \tau \cdot \Pi^Q \nabla w \, ds.$$

Integrating the line integral of the left hand side leads to

$$\int_{E_{ij}} \tau \cdot \nabla \Pi^W w \, ds \quad = \quad (\Pi^W w)(V_j) - (\Pi^W w)(V_i).$$

The edge-functionals are, per definition, bi-orthogonal to the nodal basis, thus the right hand side simplifies to

$$\int_{E_{ij}} \tau \cdot \Pi^Q \nabla w \, ds = \int_{E_{ij}} \tau \cdot \sum_{E_{i'j'}} \Psi^Q_{i'j'}(\nabla w) \varphi^Q_{i'j'} \, ds = \Psi^Q_{ij}(\nabla w).$$

Further evaluation leads to

$$\begin{aligned}
\Psi^Q_{ij}(\nabla w) &= \int\limits_{\omega_i} \int\limits_{\omega_j} f_i(y_1) f_j(y_2) \int\limits_{[y_1,y_2]} \tau \cdot \nabla w \, ds \, dy_2 dy_1 \\
&= \int\limits_{\omega_i} \int\limits_{\omega_j} f_i(y_1) f_j(y_2) \{w(y_2) - w(y_1)\} \, dy_2 dy_1 \\
&= \int\limits_{\omega_i} f_i(y_1) \, dy_1 \int\limits_{\omega_j} f_j(y_2) w(y_2) \, dy_2 - \int\limits_{\omega_i} f_i(y_1) w(y_1) \, dy_1 \int\limits_{\omega_j} f_j(y_2) \, dy_2 \\
&= (\Pi^W w)(V_j) - (\Pi^W w)(V_i).
\end{aligned}$$

The last step used $\int_{\omega_i} f_i(y) dy = 1$. The other two identities follow the same line. We have to verify

$$\int_{F_{ijk}} \nu \cdot \operatorname{curl} \Pi^Q q \, ds = \int_{F_{ijk}} \nu \cdot \Pi^V \operatorname{curl} q \, ds$$

By Stokes' theorem, the left hand side evaluates to

$$\int\limits_{F_{ijk}} \nu \cdot \operatorname{curl} \Pi^Q q \, ds = \int\limits_{\substack{E_{ij}+E_{jk} \\ +E_{ki}}} \tau \cdot \Pi^Q q \, ds = \Psi^Q_{ij}(q) + \Psi^Q_{jk}(q) + \Psi^Q_{ki}(q).$$

Again, by bi-orthogonality, the right hand side simplifies to $\Psi^V_{ijk}(\operatorname{curl} q)$, and, using definitions, Stokes' theorem, and $\int_{\omega_i} f_i \, dx = 1$, we conclude

$$\begin{aligned}
\Psi^V_{ijk}(\operatorname{curl} q) &= \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k} f_i(y_1) f_j(y_2) f_k(y_3) \int\limits_{[y_1,y_2,y_3]} \nu \cdot \operatorname{curl} q \, ds \; dy_3 dy_2 dy_1 \\
&= \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k} f_i(y_1) f_j(y_2) f_k(y_3) \int\limits_{\substack{[y_1,y_2]+[y_2,y_3]\\+[y_3,y_1]}} \tau \cdot q \, ds \; dy_3 dy_2 dy_1 \\
&= \Psi^Q_{ij}(q) + \Psi^Q_{jk}(q) + \Psi^Q_{ki}(q).
\end{aligned}$$

The last identity is left to the reader. □

Remark 2. *In the proof of Theorem 1, we have observed the discrete integration rules*

$$\begin{aligned}
\Psi^Q_{ij}(\nabla w) &= \Psi^W_j(w) - \Psi^W_i(w), \\
\Psi^V_{ijk}(\operatorname{curl} q) &= \Psi^Q_{ij}(q) + \Psi^Q_{jk}(q) + \Psi^Q_{ki}(q), \\
\Psi^S_{ijkl}(\operatorname{div} v) &= \Psi^V_{ijk}(v) + \Psi^V_{ijl}(v) + \Psi^V_{ikl}(v) + \Psi^V_{jkl}(v).
\end{aligned}$$

Lemma 3 (L_2-stability)**.** *The operators Π^W, Π^Q, Π^V, and Π^S are well defined on L_2. Their norms are independent of the local mesh size.*

Proof. The L_2-norm of finite element functions is equivalent to a properly scaled l_2 norm of nodal values (see [1] for the vector valued elements):

$$\begin{aligned}
\|w_h\|_0^2 &\simeq \sum_{V_i \in \mathcal{V}} h_i^3 w_h(V_i)^2 \\
\|q_h\|_0^2 &\simeq \sum_{E_{ij} \in \mathcal{E}} h_i \Big(\int_{E_{ij}} \tau \cdot q_h \, ds \Big)^2 \\
\|v_h\|_0^2 &\simeq \sum_{F_{ijk} \in \mathcal{F}} h_i^{-1} \Big(\int_{F_{ijk}} \nu \cdot v_h \, ds \Big)^2 \\
\|s_h\|_0^2 &\simeq \sum_{T_{ijkl} \in \mathcal{T}} h_i^{-3} \Big(\int_{T_{ijkl}} s_h \, dx \Big)^2
\end{aligned}$$

The nodal values of $\Pi^W w$ are bounded by

$$\begin{aligned}
(\Pi^W w)(V_i) &= \Psi^W_i(w) = \int_{\omega_i} f_i(y) w(y) \, dy \\
&\leq \|f_i\|_0 \, \|w\|_{0,\omega_i} \simeq h_i^{-3/2} \|w\|_{0,\omega_i},
\end{aligned}$$

summing up and using the finite overlap of ω_i, we obtain the (well known) result for Π^W. Next, we check stability for the edge-interpolation operator by using its nodal values

$$\int\limits_{E_{ij}} \tau \cdot (\Pi^Q q)\, ds = \Psi_{ij}^Q(q) = \int\limits_{\omega_i}\int\limits_{\omega_j} f_i(y_1) f_j(y_2) \int\limits_{[y_1,y_2]} \tau \cdot q\, ds\; dy_1 dy_2.$$

Transforming the line $[y_1, y_2]$ to the unit interval, i.e.,

$$\int_{[y_1,y_2]} \tau \cdot q\, ds = \int_0^1 (y_2 - y_1) \cdot q(y_1 + \sigma(y_2 - y_1))\, d\sigma,$$

allows to switch the order of integration, and the estimate $|y_2 - y_1| \preceq h_i$ leads to

$$\Psi_{ij}^Q(q) \preceq h_i \int\limits_0^1 \int\limits_{\omega_i}\int\limits_{\omega_j} |f_i(y_1)|\, |f_j(y_2)|\, |q(y_1 + \sigma(y_2 - y_1))|\, dy_2 dy_1 d\sigma.$$

In the following, we distinguish the two cases $\sigma > 1/2$ and $\sigma \leq 1/2$, and evaluate on the first. By means of the linear transformation

$$L : \omega_j \to [\omega_i, \omega_j] \;:\; y_2 \to y_1 + \sigma(y_2 - y_1),$$

Cauchy-Schwartz w.r.t. ω_j, and transformation rules, the upper half of the integral is bounded by

$$\begin{aligned}
& h_i \int_{1/2}^1 |f_i(y_1)|\, \|f_j\|_{0,\omega_j}\, \|q(L\cdot)\|_{0,\omega_j}\, dy_1 d\sigma \\
= \;& h_i \int_{1/2}^1 |f_i(y_1)|\, \|f_j\|_{0,\omega_j}\, \sigma^{-3/2} \|q(\cdot)\|_{0,L(\omega_j)}\, dy_1 d\sigma \\
\preceq \;& h_i \int_{1/2}^1 |f_i(y_1)|\, \|f_j\|_{0,\omega_j}\, \|q\|_{0,[\omega_i,\omega_j]} dy_1 d\sigma \\
\preceq \;& h_i \|f_i\|_{L_1} \|f_j\|_0 \|q\|_{0,[\omega_i,\omega_j]} \\
\preceq \;& h_i^{-1/2} \|q\|_{0,[\omega_i,\omega_j]}.
\end{aligned}$$

The case $\sigma \leq 1/2$ is symmetric, which completes the bound for Π^Q. The boundedness of Π^V and Π^S are similar. □

Lemma 4 (Consistency)**.** *Let all functions f_i be consistent up to order p. The operators Π^V, Π^Q, Π^V, and Π^S require $p \geq 0$, $p \geq 1$, $p \geq 2$, and $p \geq 3$, respectively, to preserve constants on patches.*

Proof. We show that for constant arguments the local averaging functionals coincide with the nodal interpolation operators. This is clear for Π^W. Now, consider Π^Q applied to the constant q:

$$\begin{aligned}
\Psi_{ij}^Q(q) &= \int\limits_{\omega_i}\int\limits_{\omega_j} f_i(y_1) f_j(y_2) \int\limits_{[y_1,y_2]} \tau \cdot q \, ds\, dy_1 dy_2 \\
&= \int\limits_0^1 \int\limits_{\omega_1}\int\limits_{\omega_2} f_i(y_1) f_j(y_2)(y_2 - y_1) \cdot q \, dy_2 dy_1 d\sigma \\
&= \int\limits_{\omega_1} f_i(y_1) dy_1 \int\limits_{\omega_2} f_j(y_2)(y_2 \cdot q)\, dy_2 - \int\limits_{\omega_1} f_i(y_1)(y_1 \cdot q) dy_1 \int\limits_{\omega_2} f_j(y_2), dy_2 \\
&= 1\,(V_2 \cdot q) - (V_1 \cdot q)\, 1 = \int\limits_{[V_1,V_2]} \tau \cdot q \, ds.
\end{aligned}$$

The argument is similar for Π^V. Now, the transformation to the unit triangle $\hat{F}$ leads to a second order polynomial with respect to the vertices y_1, y_2, and y_3:

$$\begin{aligned}
\Psi_{ijk}^V(v) &= \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k} f_i(y_1) f_j(y_2) f_k(y_3) \int\limits_{[y_1,y_2,y_3]} \nu \cdot v \, ds\, dy_3 dy_2 dy_1 \\
&= \int\limits_{\hat{F}}\int\limits_{\omega_1}\int\limits_{\omega_2}\int\limits_{\omega_3} f_i f_j f_k \left[(y_2 - y_1) \times (y_3 - y_1)\right] \cdot v \, dy_3 dy_2 dy_1 d\sigma \\
&= \left[(V_2 - V_1) \times (V_3 - V_1)\right] \cdot v = \int\limits_{[V_1,V_2,V_3]} \nu \cdot v \, ds.
\end{aligned}$$

Finally, the transformation of integrals for the volume element leads to the third order polynomial $\det\{y_2 - y_1, y_3 - y_1, y_4 - y_1\}$. $\square$

Together, L_2-stability and consistency give approximation:

Theorem 5 (L_2-approximation)**.** *Define for each tetrahedron $T = T_{ijkl}$ the smallest patch of elements $\widetilde{T}$ containing $[\omega_i, \omega_j, \omega_k, \omega_l]$. Let the order p be large enough according to Lemma 4. Then the following approximation estimates are valid:*

$$\begin{aligned}
\|w - \Pi^W w\|_{0,T} &\preceq h_T\, |w|_{1,\widetilde{T}} \\
\|q - \Pi^Q q\|_{0,T} &\preceq h_T\, |q|_{1,\widetilde{T}} \\
\|v - \Pi^V v\|_{0,T} &\preceq h_T\, |v|_{1,\widetilde{T}} \\
\|s - \Pi^S s\|_{0,T} &\preceq h_T\, |s|_{1,\widetilde{T}}.
\end{aligned}$$

Proof. We state the estimate for W, the others are completely identic. Define the mean value $\overline{w} = |\widetilde{T}|^{-1} \int_{\widetilde{T}} w \, dx$, and use consistency, L_2-continuity, and Friedrichs' inequality:

$$\begin{aligned} \|w - \Pi^W w\|_{0,T} &= \|(I - \Pi^W)(w - \overline{w})\|_{0,T} \\ &\preceq \|(w - \overline{w})\|_{0,\widetilde{T}} \\ &\preceq h_T \, |w|_{1,\widetilde{T}}. \end{aligned}$$

□

Corollary 6 (Approximation in semi-norm)**.** *The interpolation operators fulfill the following approximation properties with respect to the semi-norms:*

$$\begin{aligned} \|\nabla(w - \Pi^W w)\|_{0,T} &\preceq h_T |\nabla w|_{1,\widetilde{T}} && \textit{for } p \geq 1 \\ \|\operatorname{curl}(q - \Pi^Q q)\|_{0,T} &\preceq h_T |\operatorname{curl} q|_{1,\widetilde{T}} && \textit{for } p \geq 2 \\ \|\operatorname{div}(v - \Pi^V v)\|_{0,T} &\preceq h_T |\operatorname{div} q|_{1,\widetilde{T}} && \textit{for } p \geq 3 \end{aligned}$$

Proof. The estimates follow directly from the commuting diagram property and from the approximation in L_2. We show the case for $H(\text{curl})$, the others are identic:

$$\|\operatorname{curl}(q - \Pi^Q q)\|_{0,T} = \|(I - \Pi^V) \operatorname{curl} \, q\|_{0,T} \preceq h_T |\operatorname{curl} \, q|_{1,T}.$$

□

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