Nested Multigrid Methods for the Fast Numerical Computation of 3D Magnetic Fields

M. Schinnerl, J. Schöberl, and M. Kaltenbacher

***Abstract*— This paper deals with the numerical solution of static as well as transient 3D magnetic field problems. Thereby, a finite element method (FEM) with the magnetic vector potential as field variable and a discretisation with edge elements is used. For the efficient solution of the obtained matrix equation system a nested geometrical multigrid solver (MG) is presented, which reduces the solution time considerably. Numerical simulations demonstrate the superiority of the proposed method versus conventional solving strategies.**

***Index Terms*— Multigrid, finite element methods, edge elements, magnetostatics.**

I. Introduction

In the last years edge elements have been widely used for solving 3D magnetostatic as well as eddy current problems [1], [2]. The main severity to simulate 3D magnetic fields with edge elements is the large computational effort for the arising matrix equation system especially in the transient and nonlinear case. It is state of the art to use iterative methods like the conjugate gradient method with incomplete Cholesky factorization as preconditioner (ICCG). However, the number of necessary iterations strongly grows with the number of unknowns [3].
In this paper a nested multigrid solver is presented which is considerably faster than the conventional solution strategies. This is attained by constructing a hierarchy of finite edge-element spaces and by choosing appropriate smoothing procedures for each of these spaces.

II. 3D Finite Element Analysis with Edge Elements

Introducing the magnetic vector potential $\vec{A}$ by

$$\vec{B} = \nabla \times \vec{A}, \tag{1}$$

the Maxwell equations in the static case can be formulated as

$$\nabla \times (\frac{1}{\mu} \nabla \times \vec{A}) \;=\; \vec{J_0} \;\text{ in }\; \Omega \tag{2}$$

with boundary conditions

$$(\nu \nabla \times \vec{A}) \times \vec{n} \;=\; 0 \;\text{ on }\; \Gamma_H \tag{3}$$

$$\vec{n} \times \vec{A} \;=\; \vec{0} \;\text{ on }\; \Gamma_B, \tag{4}$$

where μ denotes the permeability, $\vec{J_0}$ the current density, Ω the domain, Γ_H the boundary where $\vec{H} \times \vec{n}$ is specified and Γ_B the boundary where $\vec{B} \cdot \vec{n}$ is specified.
By using edge elements for the interpolation of the vector potential, it is approximated as

$$\vec{A} \approx \sum_{k=1}^{N_e} a_k \vec{N}_k \,, \tag{5}$$

where N_e is the number of edges in the finite element mesh, $\vec{N}_k$ the edge shape function associated with the k-th edge, and a_k the corresponding degree of freedom, namely the line integral of the magnetic vector potential along the k-th edge. The application of Galerkin techniques to (2) - (4) results in the following system of algebraic equations

$$\begin{aligned} a_k \int_\Omega (\frac{1}{\mu} \nabla \times \vec{N}_k) \cdot (\nabla \times \vec{N}_i) d\Omega \\ = \int_\Omega \vec{J_0} \cdot \vec{N}_i d\Omega + \int_{\Gamma_h} \vec{K} \cdot \vec{N}_i d\Gamma \\ i = 1, 2, ..., N_e. \end{aligned} \tag{6}$$

In matrix form (7) can be rewritten as

$$\mathbf{P}\{a\} = \{F\}, \tag{7}$$

where $\mathbf{P}$ denotes the permeability matrix, $\{F\}$ the right hand side and $\{a\}$ the vector of the unknowns at the edges. The matrix $\mathbf{P}$ is only positive semi-definite and the zero eigenvalues correspond to the curl free gradient functions spanned by the edge shape functions [4].

III. Multigrid Solution Scheme

In contrast to standard FE techniques the multigrid method is not based on a fixed FE mesh, which describes the unknown field variable accurate enough. By using multigrid techniques, first a very coarse discretisation T_1 of the field problem is chosen. Then the elements of T_1 are dissected and thereby a finer discretisation T_2 is generated. The dissection of one tetrahedron element of T_1 into 8 tetrahedron elements of T_2 is shown in Fig. 1. This refinement

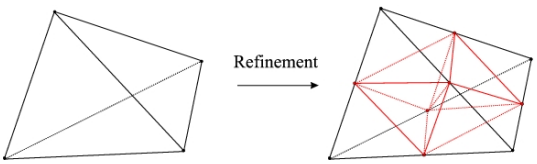

Fig. 1. Refinement of a tetrahedron element

process can include either all elements (uniform refinement)

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or only a part of the elements (adaptive refinement). The dissection of the mesh is repeated, until a discretisation T_l is obtained, which is able to describe the magnetic field accurate enough. By repeating the refinement, a hierarchy of FE discretisations $T_1, ..., T_l$ is generated, and for each of the FE meshes T_q, $q = 1, 2, ..., l$ the equation system (7), which is abbreviated as

$$\mathbf{P}_q a_q = f_q \tag{8}$$

can be assembled. The aim of the multigrid technique is the solution of (8) at the finest level l in a very fast way. Additionally, an interpolation operator I_q^{q+1} is defined, which transfers an arbitrary FE edge function ξ_q at level q into an equivalent edge function ξ_{q+1} at level $q+1$, i.e. the condition

$$\xi_{q+1} = I_q^{q+1} \xi_q \tag{9}$$

holds. Vice versa the restriction of the edge function ξ_{q+1} to the coarser grid q is given by

$$\xi_q = I_{q+1}^{q} \xi_{q+1}, \tag{10}$$

whereby I_{q+1}^{q} denotes the restriction operator. At each level q smoothing operators G_q are defined, which must be adapted to the magnetic problem. Using the components I_q^{q+1}, I_{q+1}^{q} and G_q the classical two-grid algorithm, as the simplest exponent of the multigrid methods, can be defined.

A. Classical two-grid algorithm

$a_{q,j,s}$ denotes the approximation of the solution a at the level q, the current multigrid iteration j and the status of the current iteration s. In the two-grid method the multigrid hierarchy only consists of one coarse grid $q-1$ and one finer grid q. The aim of this method is the solution of (8) at level q.

Given: Approximation $a_{q,j}$ after j two-grid iterations
1. ν_1 smoothing operations at the finest level q

$$a_{q,j,1} = (G_q)^{\nu_1} a_{q,j} \tag{11}$$

2. Calculation of the defect

$$d_q = f_q - \mathbf{P}_q x_{q,j,1} \tag{12}$$

and restriction of the defect to the coarse grid

$$d_{q-1} = I_q^{q-1} d_q. \tag{13}$$

3. Solution of the coarse grid system

$$\mathbf{P}_{q-1} w_{q-1} = d_{q-1}. \tag{14}$$

4. Prolongation of w_{q-1} to the finer grid

$$w_q = I_{q-1}^{q} w_{q-1} \tag{15}$$

and correction of the smoothed approximation $a_{q,j,1}$

$$a_{q,j,2} = a_{q,j,1} + w_q. \tag{16}$$

5. ν_2 smoothing operations at the level q

$$a_{q,j,3} = (G_q)^{\nu_2} a_{q,j,2} \tag{17}$$

MG iteration stop: $a_{q,j+1} = a_{q,j,3}$

The presented two-grid algorithm leads to very good approximations of a after a small number of iterations. Thereby, at each two-grid iteration only $\nu 1 + \nu 2$ smoothing operations and a direct solution at the level $q-1$ must be performed. Nevertheless, in many cases the solution process of (14) at the coarser grid $q-1$ consumes too much time and memory. In order to avoid this difficulty the two-grid algorithm is recursively restarted for (14). Repeating this process for all available grids a MG V-cycle is generated which is displayed in Fig. 2. Thereby, at each level only smoothing operations are necessary, and at the coarsest grid $q = 1$ a linear equation system of the form $\mathbf{P}_1 w_1 = d_1$ must be solved with a conventional direct or iterative solver. Since at this grid the number of unknowns is generally low, this solution process is not very time and memory consuming. Additionally, it can be shown, that MG methods have "optimal complexity", that means, that the number of necessary calculation operations and the memory consumption increase only linear with the number of unknowns at the finest level l [5]. Thereby, the performance of the MG algorithm depends on the appropriate choice of the grid transfer operators and the smoothing techniques, which must be adapted to the FE discretisation with edge tetrahedron elements [4].

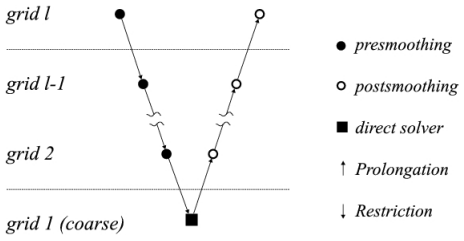

Fig. 2. MG solution algorithm (V-Cycle)

B. Choosing the grid transfer operators

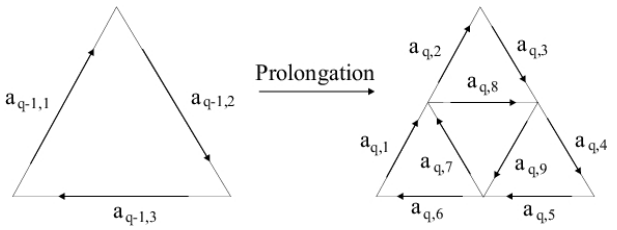

Fig. 3. Prolongation of the edge degrees of freedom from level $q-1$ to level q

In order to determine the grid transfer operators between the levels $q-1$ and q, the refinement of a face F^{q-1} of an

edge tetrahedron element on the level $q-1$ is analyzed. By dissecting the tetrahedron at the level $q-1$ into 8 tetrahedrons at the level q, the area of F^{q-1} is replaced by 4 new faces $F_1^q, .., F_4^q$ (Fig. 3). A suitable prolongation I_{q-1}^q must guarantee that the magnetic flux Φ across F^{q-1} and $F_1^q + .. + F_4^q$ must be equal, i.e. the condition

$$\int_{F^{q-1}} (\nabla \times \vec{A}) \cdot d\vec{F} = \sum_{k=1}^{4} \int_{F_k^q} (\nabla \times \vec{A}) \cdot d\vec{F} \qquad (18)$$

must hold. Using Stoke's law, (18) can be rewritten as the integral over the edges

$$\oint_{\partial F^{q-1}} \vec{A} \cdot d\vec{s} = \sum_{k=1}^{4} \oint_{\partial F_k^q} \vec{A} \cdot d\vec{s}, \qquad (19)$$

or, by exploiting the degrees of freedom of the FE formulation (Fig. 3), as

$$\begin{aligned} a_{q-1,1} + a_{q-1,2} + a_{q-1,3} = a_{q,1} - a_{q,7} + a_{q,6} + a_{q,2} \\ +a_{q,3} - a_{q,8} + a_{q,4} + a_{q,5} - a_{q,9} + a_{q,7} + a_{q,8} + a_{q,9}. \end{aligned} \qquad (20)$$

The transfer operator

$$I_{q-1}^q = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 & -.25 & -.25 & .25 \\ 0 & 0 & .5 & .5 & 0 & 0 & .25 & -.25 & -.25 \\ 0 & 0 & 0 & 0 & .5 & .5 & -.25 & .25 & -.25 \end{pmatrix}^T$$

fulfills the requirements of the flux conservation, which can be proofed by inserting I_{q-1}^q into (20). The restriction operator I_{q+1}^q is chosen to be the transposed to I_q^{q+1}, i.e.

$$I_{q+1}^q = (I_q^{q+1})^T. \qquad (21)$$

C. Smoothing operators

The permeability matrix $\mathbf{P}$ is only positive semi-definite and therefore, a positive definite matrix $\mathbf{M}$ is added in order to achieve regularity of the resulting matrix

$$\tilde{\mathbf{P}} = \mathbf{P} + \epsilon \mathbf{M}. \qquad (22)$$

Thereby, ϵ must be chosen very small to prevent calculation errors. This small ϵ deteriorates the convergence behavior of standard Gauss-Seidel smoothers [6]. In order to achieve an improved smoother, which is independent from the size of ϵ, the mathematical properties of the edge element discretization must be analyzed in more detail. It is possible to split the edge element space V_{edge} into two parts, i.e.

$$V_{edge} = V_{edge,0} + V_{edge,1}. \qquad (23)$$

$V_{edge,0}$ denotes the space of the gradients of the nodal shape functions spanned in the same mesh as the edge elements, and $V_{edge,1}$ the orthogonal complement to $V_{edge,0}$ [4]. By considering all edges which share a common node, an overlapping block Gauss-Seidel smoother can be constructed, which is able to damp out errors also in the space $V_{edge,0}$ [7]. To specify the method we define for each of the N_V vertices V_k the connectivity matrix

$$\mathbf{R}_k = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 1 & 0 & \dots \\ \dots & \dots & & & & \end{pmatrix}.$$

It has number of edges on vertex V_k rows and N_e columns. Using this matrix, we can pick out the quadratic sub-blocks P_k of the matrix P by

$$\mathbf{P}_k = \mathbf{R}_k^T \tilde{\mathbf{P}} \mathbf{R}_k. \qquad (24)$$

Each of these small matrices have to be inverted in the preparation phase of the multigrid method. One step of the block Gauss-Seidel iteration with initial approximation $a_1 = a$ is defined as

$$a_{k+1} = a_k + \mathbf{R}_k^T \mathbf{P}_k^{-1} \mathbf{R}_k (f - \tilde{\mathbf{P}} a_k), \qquad k = 1, \dots V_k. \qquad (25)$$

We mention, that not the whole residual $f - \tilde{\mathbf{P}} a_k$ has to be computed at each step, but only the few components picked out by $\mathbf{R}_k$. Therefore, one block Gauss-Seidel step is not much more expensive then a simple Gauss-Seidel step.

D. Nested Multigrid

Using MG techniques the domain is discretised with a number of FE meshes $T_1, ..., T_l$. Since the number of unknowns at the coarsest level T_1 is generally small, a solution of (8) requires less numerical effort. The solution a_1 at this coarse level can now be prolongated by

$$a_2^{app} = I_1^2 a_1 \qquad (26)$$

to the next finer grid, where a_2^{app} is used as start approximation for the solution at T_2, and therefore the number of necessary MG iterations at this level is decreased. By repeating this process for all meshes $T_1, ..., T_l$ the number of necessary iterations at the finest grid can be reduced considerably.

IV. Numerical Results

To show the applicability of the proposed scheme the TEAM problem #20 was computed [8]. In Fig. 4 a coarse grid mesh of this 3D, nonlinear and static problem is shown. In order to show the advantages of the MG technique to large scaled problems, a uniform refinement of the coarse grids is chosen, i.e. each edge tetrahedron-element on a level $q-1$ is divided into 8 elements on the level q. Thereby, 4 hierarchical meshes are generated, where the coarsest one has 2190 unknowns and the finest one 1070000 unknowns. The problem was solved with the presented nested MG technique, whereby, in a first step, a constant relative permeability μ_r of the iron parts was chosen. In order to show the robustness of the proposed method to parameter jumps, the computation was repeated for different values of μ_r. Table I compares the number of iterations for the different meshes and permeabilities of the iron parts. As it was to be expected, the used nested multigrid algo-

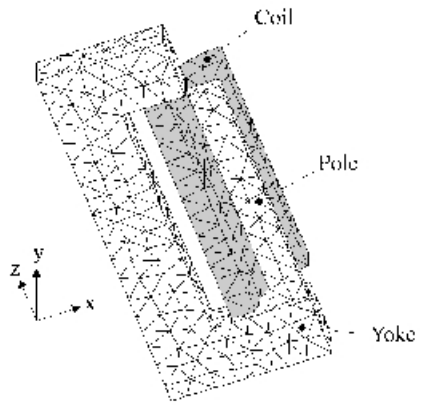

Fig. 4. Coarse FE discretisation of TEAM problem #20

TABLE I
NECESSARY MG ITERATIONS (V(1,1) CYCLE) TO REACH A RESIDUUM OF 10^{-6}

Number of edges	iterations (nested) $\mu = 1$	$\mu = 10^3$	$\mu = 10^4$
30500	9	10	10
236000	10	11	10
1900000	10	11	11

rithm is robust against parameter variations. Furthermore the number of necessary multigrid iterations only slightly increases with the number of unknowns.

To show the advantages of the proposed method, the magnetic field, using the same discretisations, has been computed applying a CG solver with adapted block-preconditioning but without MG strategies (PCCG). The normalized residual versus the number of iterations for the PCCG method and for the nested MG method on the finest mesh is shown in Fig. 5. For MG a very good conver-

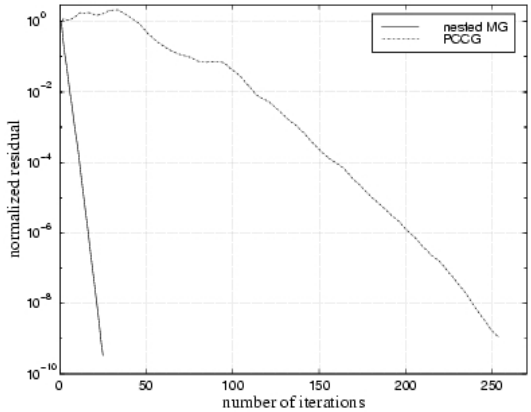

Fig. 5. Number of iterations versus normalized residual (1900000 dofs)

gence for the whole considered residual domain is received, whereas the PCCG method shows a much slower and not monoton reduction of the residuum. Table II compares the solution times for solving the matrix equation system using nested MG and PCCG on a SGI Origin 300 MHz. As expected nested MG is especially for large scaled problems considerably faster than conventional approaches.

In order to verify the calculation scheme, the TEAM problem #20 was calculated with the nonlinear permeablity given in [8]. In Table III the computed magnetic force f_{mag} to the pole was compared to measurement. Thereby, for all excitations Θ of the coil, a good agreement between measurement and simulation is achieved.

TABLE II
COMPARISON OF SOLUTION TIMES

number of edges	nested MG (seconds)	precond. CG (seconds)
30500	7	19
236000	95	362
1900000	870	9860

TABLE III
MAGNETIC FORCE f_z ACTING AT THE CENTER POLE

$\Theta(A)$	F_z (N) simulation	measurement [8]
1000	7.83	8.1
3000	53.7	54.4
4500	73.2	75.0
5000	77.5	80.1

V. CONCLUSION

In this paper we have presented a technique for the calculation of 3D magnetic field problems, based on a hierarchy of FE discretisations. Thereby, the different components, like smoothing operations, prolongations and nested techniques, are explained and analyzed. By computing the TEAM problem #20, the advantages of the presented scheme are demonstrated.

REFERENCES

[1] K. Fujiwara, " 3-D Magnetic Field Computation using Edge Elements," in *5th International IGTE Symposium*, 1992, pp. 185–212.

[2] J. Nédélec, "Mixed finite elements in R^3," *Numer.Math.*, pp. 315–341, 1980.

[3] O. Biró, K. Preis, and K.R. Richter, "On the Use of the Magnetic Vector Potential in the Nodal and Edge Finite Element Analysis of 3D Magnetostatic Problems," *IEEE Transactions on Magnetics*, vol. 32, no. 3, pp. 651–654, May 1996.

[4] R. Hiptmair, "Multigrid method for Maxwell's equations," *SIAM J. Numer. Anal.*, vol. 36, no. 1, pp. 204–225, May 1996.

[5] M. Jung and U. Langer, "Applications of multilevel methods to practical problems," *Surv. Math. Ind.*, vol. 1, pp. 217–257, 1991.

[6] H.R. Schwartz, *Numerische Mathematik*, B.G. Teubner, Stuttgart, 1 edition, 1993.

[7] D.N. Arnold, R.S. Falk, and R. Winther, "Multigrid in H(div) and H(curl)," 1998, Department of Mathematics, Penn State University.

[8] N. Takahashi, T. Nakata, and H. Morishige, "Summary of results for problem 20 (3-d static force problem)," *COMPEL*, vol. 14, no. 2/3, pp. 57–75, 1985.