Efficient Contact Solvers Based on Domain Decomposition Techniques

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Abstract

This paper deals with the construction of efficient algorithms for the solution of the finite dimensional constrained minimization problem arising from the finite element discretization of contact problems.

Dualization techniques have been used to decrease the problem size from the large number of unknowns in the domain to the much smaller number of inequalities at the boundary. The disadvantage of this direct Schur-complement approach is the need of the inversion of the stiffness matrix.

Domain decomposition techniques meet very similar requirements. A global boundary value problem is decoupled into local subproblems, and one interface problem at the (coupling) boundary. Two complementary approaches are the Dirichlet method and the Neumann method. The first one requires preconditioners for local Dirichlet problems and for the interface problem in $H^{+1/2}$, and extension operators from the boundary into the domain. The second one needs preconditioners for local Neumann problems, and for the interface problem in $H^{-1/2}$. Efficient multi-level algorithms for all components are available in literature.

In this paper it is shown how to use exactly these components for the construction of solvers for contact problems. New results for the analysis of convergence are presented. At least at uniformly refined meshes, we can prove optimal time complexity. Numerical results show high efficiency also on adaptively refined 3d meshes.

AMS Subject Classifications: 73T05, 35J85, 65N55, 65F35, 65K10

Key words: contact problem, variational inequality, domain decomposition, preconditioning.

1 Introduction

The contact problem is an important problem in computational mechanics. Elastic bodies are deformed due to volume and surface forces, but the bodies should not penetrate each other. A simplified problem is the Signorini problem, where one body should not penetrate a given, rigid obstacle. Both lead to unilateral boundary conditions, the contact conditions. We refer to [24], [7], [5], [19], [8], [18], [13] for mathematical modeling, analysis and finite element discretization.

In this paper we are interested in fast, iterative algorithms for solving the arising finite dimensional constrained minimization problem.

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There are classical iterative methods like point projection methods and point over-relaxation methods [8]. These methods suffer from slow convergence rates on fine meshes. By conjugate gradient like methods the speed of convergence can be improved [6], but still the number of necessary iterations depends on the mesh size. Multigrid methods have been successfully applied to obstacle problems with inequality constraints in the whole domain by [11], [21], [14], [15] and to Signorini's problem in [20].

In this paper we describe different realizations of the preconditioned projection method originally investigated in this context in [23]. This method is efficient, iff

- the relative condition number of the system matrix A with respect to the preconditioning matrix C is small,
- the operation $C^{-1} \times v$ is fast executable,
- the projection P with respect to the C energy norm onto the feasible set K is fast computable.

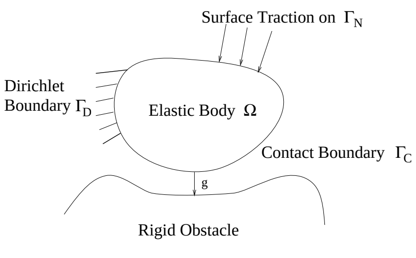
Multigrid preconditioners as well as domain decomposition preconditioners have been developed to satisfy the first two requirements quite well. We discuss, how to combine standard multigrid and domain decomposition components to fulfill all three requirements. This enables us to reuse available efficient implementations. The basic idea is an approximative decoupling of the many equations in the domain from the still a lot, but much less inequalities on the boundary. The decoupling is done by two approaches dual to each other. One is connected with Dirichlet domain decomposition, the other one with Neumann domain decomposition techniques.

The theoretical estimates prove optimal time and memory complexity on uniformly hierarchical refined meshes. Numerical experiments indicate also efficient behavior on adaptively refined meshes.

The rest of the paper is organized as follows. In Section 2 the problem and discretization is given. Section 3 shortly present domain decomposition techniques, Sections 4 and 5 give the two realizations of the projection algorithm. Finally, in Section 6 numerical results are presented.

2 Problem Description

The problem of Signorini is sketched in the picture



and formulated in classical form as follows. The domain Ω is supposed to be bounded in $\mathbf{R}^d$, $d = 2$ or 3 with a Lipschitz-continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ with $meas(\Gamma_D) \neq 0$ and $meas(\Gamma_C) \neq 0$. Further, n is the unit normal vector, u is the (unknown) displacement field, $e(u)$ is the strain operator $e(u) = 0.5(\nabla u + \nabla u^T)$, D is the tensor of elastic coefficients and σ is the (unknown) stress tensor. Given functions are the volume force f, the surface traction g and the gap g to the rigid obstacle. The equations with boundary conditions read as

$$\begin{array}{rcll} \sigma - De(u) & = & 0 & \text{in } \Omega, \\ -\operatorname{div} \sigma & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma_D, \\ \sigma n & = & g & \text{on } \Gamma_N, \\ \sigma n - (n^T \sigma n) n & = & 0 & \text{on } \Gamma_C, \\ u^T n \leq g, \quad n^T \sigma n \geq 0, \quad (u^T n \leq g)\, (n^T \sigma n) & = & 0 & \text{on } \Gamma_C. \end{array} \tag{1}$$

The weak form of (1) is discretized by the finite element method on a mesh with totally N nodes and N_C nodes at the boundary Γ_C. We obtain the finite dimensional Constrained Minimization Problem (CMP)

$$\text{Find } u \in K: \quad J(u) = \inf_{v \in K} J(v), \quad \text{with} \quad J(v) := \frac{1}{2} v^T A v - f^T v. \tag{2}$$

The stiffness matrix A is symmetric and positive definite of dimension R^{dN}, f is the load vector. The convex set of feasible functions $K \subset V := \mathbf{R}^{dN}$ is defined by

$$K = \{ v \in V : Bv \geq g \}, \tag{3}$$

where $\geq$ is meant component-wise. The matrix B of dimension $N_C \times dN$ is defined such that for all nodes x_i on the contact boundary and for all $v \in V$ there holds

$$(Bv)_i = n(x_i)^T v(x_i), \tag{4}$$

and $g_i = g(x_i)$.

By means of the matrix B we can define the corresponding mixed form as find $u \in V$, $p \in \Lambda := (\mathbf{R}_0^+)^{N_C}$ such that

$$\begin{array}{lcrcl} Au & + & B^T p & = & f, \\ Bu & & & \geq & g, \\ & (Bu - g)^T p & & = & 0. \end{array} \tag{5}$$

The mixed form (5) includes more general discretizations than nodal inequalities. We can use different finite element spaces for the approximation of the dual variables, see [1] for Mortar techniques. The mixed form is also suited for body-body contact problems, where each restriction involves more mesh points. We will construct iterative solvers for both forms.

For the solution of CMPs one can use the preconditioned projection algorithm. Let C denote the symmetric, positive definite preconditioning matrix with the spectral bounds

$$\underline{\alpha}\, C \leq A \leq \overline{\alpha}\, C,$$

where $A > B$ ($A \geq B$) means, that $A - B$ is positive definite (positive semidefinite). Define the projection $P_C^K : V \to K$ as

$$P_C^K(u) := \underset{v \in K}{\operatorname{argmin}} \; \|v - u\|_C, \tag{6}$$

where $\|v\|_C$ denotes the energy norm $(v^T C v)^{1/2}$. Let τ be a damping parameter $0 \leq \tau \leq \overline{\alpha}^{-1}$. Then the preconditioned projection method is defined as

Choose $u^1 \in K$
for $k = 1, 2, \ldots$ let
$$u^{k+1} = P_C^K(u^k + \tau C^{-1}(f - Au^k))$$

Except for diagonal preconditioners C, the projection P_C^K itself is not explicit available, but has to be approximated by an iterative algorithm. We will achieve an efficient method, iff the inner iteration for the evaluation of the projection P_C^K is much cheaper than the same iteration applied for the projection P_A^K. We do not want to perform the inner iteration until it "converged", but it has to reduce the error by a fixed factor $\rho_P < 1$. This leads us to the

Algorithm 1 (Approximative Projection Method)

Choose an arbitrary $u^1 \in K$.
For $k = 1, 2, \ldots$ *do*
$$\tilde{u}^k = u^k + \tau C^{-1}\left(f - Au^k\right),$$
$$u^{k+1} = \tilde{P}(\tilde{u}^k).$$

The proof of the following result of convergence can be found in [23]:

Theorem 1 (Energy convergence rate estimate)
Let u^k *be the sequence generated by Algorithm 1. The relaxation parameter* τ *is chosen in the interval* $(0, 1/\overline{\alpha}]$. *The approximative projection* $\tilde{P}$ *fulfills*

$$\|\tilde{P}(\tilde{u}^k) - \tilde{u}^k\|_C^2 \leq \rho_P \|u^k - \tilde{u}^k\|_C^2 + (1 - \rho_P)\, \|P(\tilde{u}^k) - \tilde{u}^k\|_C^2, \tag{7}$$

with $\rho_P \in [0, 1)$. *Then the estimate*

$$J(u^{k+1}) \leq \rho J(u^k) + (1 - \rho)\, J(u) \tag{8}$$

holds for every $k \in \mathbf{N}$ *with the convergence rate*

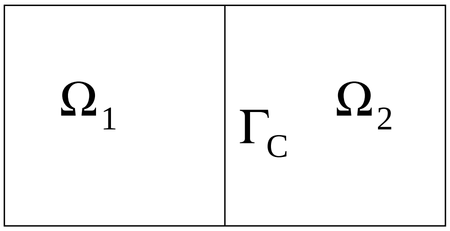
$$\rho = 1 - \frac{\tau \underline{\alpha}}{2}(1 - \rho_P) \leq 1. \tag{9}$$

The error in A*-energy norm is bounded by*

$$\|u - u^k\|_A^2 \leq 2\rho^{k-1}\left(J(u^1) - J(u)\right). \tag{10}$$

3 Domain decomposition techniques

In this section we will briefly explain some concepts of both major types of non-overlapping domain decomposition, of the Dirichlet version as well as the Neumann version. It is enough to consider the 2 sub-domain case, where the domain is split as $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and the coupling boundary is given as $\Gamma_C = \overline{\Omega}_1 \cap \overline{\Omega}_2$:



3.1 Dirichlet domain decomposition

We present the preconditioner developed in [10]. Let A be the finite element matrix from an elliptic, second order problem. We renumber the nodes by counting first the nodes at the coupling boundary $(._C)$, and than the inner nodes and nodes at natural boundaries $(._I)$. We split accordingly the space $V = \mathbf{R}^N$ into

$$V = V_C \oplus V_I$$

and the matrix

$$A = \begin{pmatrix} A_{CC} & A_{CI} \\ A_{IC} & A_{II} \end{pmatrix}.$$

The simple block diagonal preconditioner $C = \operatorname{diag}(A_{CC}, A_{II})$ would lead to a generalized condition number $\kappa(C^{-1}A)$ growing as the mesh size decreases. Let S_D denote the boundary Schur complement

$$S_D = A_{CC} - A_{CI} A_{II}^{-1} A_{IC}. \tag{11}$$

For Dirichlet domain decomposition one introduces an extension operator $E : V_C \to V$ and a basis transformation T which split into the blocks

$$E = \begin{pmatrix} I_C \\ E_I \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} I_C & 0 \\ E_I & I_I \end{pmatrix}. \tag{12}$$

The extension operator should approximate the solution of local Dirichlet problems in the sense that

$$\|E v_C\|_A \le c_E \inf_{\substack{w_I \in V_I \\ w = (v_C, w_I)}} \|w\|_A = c_E \, \|v_C\|_{S_D}$$

holds with a small constant $c_E \ge 1$. This means, that the matrix $E^T A E$ is spectrally equivalent to the Schur complement S with constants 1 and c_E^2. Further, we need a preconditioner C_I for the local Dirichlet sub-problems with matrix A_I and the so called Schur complement preconditioner C_C for the matrix $E^T A E$. We assume the spectral inequalities

$$\begin{array}{ccccc} \underline{\alpha}_C C_C & \le & E^T A E & \le & \overline{\alpha}_C C_C, \\ \underline{\alpha}_I C_I & \le & A_I & \le & \overline{\alpha}_I C_I. \end{array} \tag{13}$$

Using these components, we can define the approximative additive Dirichlet domain decomposition preconditioner by

$$C^{-1} = \begin{pmatrix} I_C & 0 \\ E_I & I_I \end{pmatrix} \begin{pmatrix} C_C^{-1} & 0 \\ 0 & C_I^{-1} \end{pmatrix} \begin{pmatrix} I_C & E_I^T \\ 0 & I_I \end{pmatrix}, \tag{14}$$

for which the spectral inequalities

$$\underline{\alpha}\, C \le A \le \overline{\alpha}\, C$$

hold with the constants

$$\underline{\alpha} = \left(1 - \sqrt{1 - c_E^{-2}}\right) \min\left\{\underline{\alpha}_C, \underline{\alpha}_I\right\} \quad \text{and} \quad \overline{\alpha} = \left(1 + \sqrt{1 - c_E^{-2}}\right) \max\left\{\overline{\alpha}_C, \overline{\alpha}_I\right\}.$$

On hierarchical refined meshes, all components are available in optimal time complexity and with bounds independent of the mesh size. As preconditioner C_I in the domain, a symmetric multigrid preconditioner [2], [17] can be used. Optimal components for the Schur complement preconditioner C_C and for the extension operator in 2D and 3D are constructed by multi-level techniques [4], see [25] for the Schur complement preconditioner and [22] for the extension operator. In [9] additional smoothing improves the constant c_E. The operation $C_C^{-1} \times v$ is implemented within $O(N_C)$ arithmetical operations.

3.2 Neumann domain decomposition

For Neumann domain decomposition one does not incorporate continuity across the interface into the continuous or finite element space, but ensure it by the linear restriction $u|_{\Omega_1} = u|_{\Omega_2}$ on Γ_C. This leads to the block diagonal matrix

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix},$$

where the blocks A^j are the sub-domain finite element matrices with Neumann boundary conditions at Γ_C. The blocks are split into boundary and inner unknowns as

$$A^{(j)} = \begin{pmatrix} A^{(j)}_{CC} & A^{(j)}_{CI} \\ A^{(j)}_{IC} & A^{(j)}_{II} \end{pmatrix}.$$

The restriction is formulated by means of the matrix

$$B = \begin{pmatrix} B^{(1)} & -B^{(2)} \end{pmatrix} \qquad \text{with} \qquad B^{(j)} = \begin{pmatrix} B^{(j)}_C & 0 \end{pmatrix}.$$

On matching grids, the block $B^{(j)}_C$ simplifies to I_C. Thus, we obtain the mixed problem

$$\begin{array}{lclcl} Au & + & B^T p & = & f, \\ Bu & & & = & 0. \end{array} \tag{15}$$

This saddle point problem can be solved by some version of inexact Uzawa algorithm, see e.g. [3]. First, a preconditioner to the matrix A is needed. Again, a multigrid preconditioner can be used. One has to be careful, because some of the blocks $A^{(j)}$ can be only semi-definite even if the global problem is elliptic. In this case, one has to work with projections to the rigid body motions. The second preconditioner is needed for the Neumann Schur-complement

$$S_N = BA^{-1}B^T = \sum_{j=1}^{2} B^{(j)}[A^{(j)}]^{-1}[B^{(j)}]^T = \sum_{j=1}^{2} B^{(j)}_C [S^{(j)}]^{-1}[B^{(j)}_C]^T \tag{16}$$

with sub-domain Schur complements $S^{(j)} = A^{(j)}_{CC} - A^{(j)}_{CI}[A^{(j)}_{II}]^{-1}A^{(j)}_{IC}$.

Optimal preconditioners for the Neumann Schur-complement S_N are constructed very recently in [16].

It is interesting to note, that due to the inner iterations in the contact algorithm, the required Schur complement preconditioners exchange.

4 Projection Algorithms based on Dirichlet domain decomposition

In this section we will apply the Dirichlet DD preconditioner for the projection method. The first two requirements, namely condition numbers independent of the mesh-size and fast execution of the preconditioning operation are fulfilled for this preconditioner. To construct the projection we use the basis transformation matrix T of (12) and express the solution u by

$$u = T\hat{u}.$$

Therefore, $\hat{u}$ is the solution of the CMP

$$\inf_{\hat{v}\in\hat{K}} \hat{J}(\hat{v})$$

with $\hat{K} = T^{-1}K$ and

$$\hat{J}(\hat{v}) = J(T\hat{v}) = \frac{1}{2}\hat{v}^T \underbrace{T^T A T}_{\hat{A}} \hat{v} - \underbrace{f^T T}_{\hat{f}^T} \hat{v}.$$

If C is a preconditioner for A, then also $\hat{C} = T^T C T$ is one for $\hat{A}$ with the same bounds. For the DDD-preconditioner (14), the transformed $\hat{C}$ has the block diagonal structure

$$\hat{C} = \begin{pmatrix} C_C & 0 \\ 0 & C_I \end{pmatrix}.$$

Because v_I can be chosen arbitrarily in a linear space, the set $\hat{K}$ reduces to K:

$$\hat{K} = T^{-1}K = \left\{ \begin{pmatrix} I_C & 0 \\ -E_I & I_I \end{pmatrix} \begin{pmatrix} v_C \\ v_I \end{pmatrix} : B_C v_C \geq g \right\} = K.$$

Now, we apply the approximative projection method to the transformed system:

$$\hat{u}^{k+1} = \tilde{P}_K^{\hat{C}} \left(\hat{u}^k + \tau \hat{C}^{-1} \left(\hat{f} - \hat{A}\hat{u}^k \right) \right). \tag{17}$$

We use the abbreviation

$$\tilde{u} = \hat{u}^k + \tau \hat{C}^{-1} \left(\hat{f} - \hat{A}\hat{u}^k \right).$$

To apply Theorem 1 the approximative projection $\tilde{P}_K^{\hat{C}}$ has to fulfill

$$\|\tilde{P}_K^{\hat{C}}(\tilde{u}) - \tilde{u}\|_{\hat{C}}^2 \leq (1-\rho_P) \|P_K^{\hat{C}}(\tilde{u}) - \tilde{u}\|_{\hat{C}}^2 + \rho_P \|\hat{u}^k - \tilde{u}\|_{\hat{C}}^2, \tag{18}$$

where $P_K^{\hat{C}}$ is the exact projection with respect to the $\hat{C}$ inner product. Because the inner product matrix $\hat{C}$ is block diagonal, and the restrictions involve unknowns only on the contact boundary, the projection reduces to

$$\tilde{P}_K^{\hat{C}}(\tilde{u}) = \left(\tilde{P}_{K_C}^{C_C}(\tilde{u}_C), \tilde{u}_I \right).$$

Inequality (18) is implied by the corresponding inequality for the boundary projection

$$\|\tilde{P}_{K_C}^{C_C}(\tilde{u}_C) - \tilde{u}_C\|_{C_C}^2 \leq (1-\rho_P) \|P_{K_C}^{C_C}(\tilde{u}_C) - \tilde{u}_C\|_{C_C}^2 + \rho_P \|\hat{u}_C^k - \tilde{u}_C\|_{C_C}^2. \tag{19}$$

By means of the quadratic functional

$$J_C(v_C) := \frac{1}{2} v_C^T C_C v_C - \tilde{u}_C^T C_C v_C \tag{20}$$

equation (19) can be written as

$$J_C\left(\tilde{P}_{K_C}^{C_C}(\tilde{u}_C)\right) \leq (1-\rho_P)\, J_C\left(P_{K_C}^{C_C}(\tilde{u}_C)\right) + \rho_P\, J_C\left(\hat{u}_C^k\right). \tag{21}$$

This problem can be solved by n steps of the projection method with trivial preconditioner I_C:

$w^0 = \hat{u}_C^k$
for $i = 0, \ldots, n-1$ do
$\quad w^{i+1} = P_{K_C}^{I_C}\left(w^i + \tau_i C_C(u_C^k - w^i)\right)$
$\tilde{P}_{K_C}^{C_C}(\tilde{u}_C) := w^n$

If we chose the optimal damping parameter $\tau_i = \lambda_{\max}(C_C)^{-1}$ and $n \geq c\kappa(C_C)$, than we obtain from Theorem 1, that (21) holds with

$$\rho_P \leq (1 - 0.5\kappa(C_C)^{-1})^{c\kappa(C_C)} \leq e^{-c/2} < 1.$$

Now, we can state the whole algorithm in nodal coordinates:

Algorithm 2 (Dirichlet Projection Method)

Choose an arbitrary $u^1 \in K$.
For $k = 1, 2, \ldots$ *do*

$\quad w_I = C_I^{-1}(0\ I_I)(f - Au^k)$
$\quad g_C = C_C u_C^k + \tau E^T(f - Au^k)$
$\quad w_C^0 = u_C^k$
$\quad$ *for* $j = 0, \ldots, n-1$ *do*
$\quad\quad w_C^{j+1} = P_{K_C}^{I_C}\left(w_C^j + \tau_i(g_C - C_C w_C^j)\right)$
$\quad u^{k+1} = u^k + \tau(0\ I_I)^T w_I + E w_C^n$

Theorem 2
Algorithm 2 has optimal time complexity on uniformly refined hierarchical meshes. This means, the time complexity for reducing the iteration error by a factor of ε*, is* $O(N|\log\varepsilon|)$ *independent of the number of levels.*

Proof: The computational costs are $n_o(c_o + n_i c_i)$, with the number of outer and inner iterations n_o and n_i, respectively, the costs c_o in the outer loop and the costs c_i per inner loop. Because $\kappa(C^{-1}A) = O(1)$, and $\rho_P < 1$ independent of the level, $n_o = O(|\log\varepsilon|)$. Because components of optimal time complexity are used, $n_o = O(N) = O(h^{-d})$ and $n_i = O(N_C) = O(h^{-d+1})$. The condition number $\kappa(C_C)$ and thus n_i is $O(h^{-1})$. Summing up, we get the total cost $O(\log\varepsilon(h^{-d} + h^{-1}h^{-d+1})) = O(N\log\varepsilon)$.

5 Projection Algorithms based on Neumann domain decomposition

The mixed form (5) is best suited for the solver based on Neumann domain decomposition techniques. By the exact elimination of the primal variable u, one obtains the equivalent, dual form

$$p = \operatorname*{argmin}_{q \in \Lambda} J^*(q) \tag{22}$$

with the dual functional

$$J^*(q) = \frac{1}{2}q^T BA^{-1}B^T q - q^T(BA^{-1}f - g). \tag{23}$$

It has been suggested to use quadratic programming methods for the minimization of this much smaller problem of dimension N_C. But, this functional requires the operation A^{-1}, which is not fast available.

To overcome this difficulty, we use the Augmented Lagrangian technique. We add a convex function in v and q the minimum of which in v equals 0 for every fixed q. We assume, that the preconditioner C for the matrix A is scaled such that $\lambda_{\min}(C^{-1}A) > 1$. Then $C^{-1} - A^{-1}$ is positive definite. So, we obtain the equivalent problem

$$(u,p) = \underset{\substack{v\in V\\ p\in\Lambda}}{\operatorname{argmin}} \ \frac{1}{2}\,\|Av + B^T q - f\|_{C^{-1}-A^{-1}} + J^*(q). \tag{24}$$

The first nice feature is, that the matrix $\mathcal{A}$ and the vector $\mathcal{F}$ of the quadratic form, which evaluate to

$$\mathcal{A} = \begin{pmatrix} AC^{-1}A - A & (AC^{-1} - I)B^T \\ B(C^{-1}A - I) & BC^{-1}B^T \end{pmatrix} \qquad \text{and} \qquad \mathcal{F} = \begin{pmatrix} (AC^{-1} - I)f \\ BC^{-1}f - g \end{pmatrix} \tag{25}$$

are fast applicable or computable, respectively. It was observed in [3], that the matrix $\mathcal{A}$ is spectrally equivalent to the block preconditioning matrix $\widetilde{\mathcal{C}}$

$$\widetilde{\mathcal{C}} = \begin{pmatrix} A - C & 0 \\ 0 & BA^{-1}B^T \end{pmatrix}$$

with condition number $\kappa(\widetilde{\mathcal{C}}^{-1}\mathcal{A})$ asymptotically as good as the condition number $\kappa(C^{-1}A)$. The block $A - C$ cancels out in the first row of $\mathcal{A}$ as well as in $\mathcal{F}$. In [23] similar estimates are given for the block diagonal matrix

$$\begin{pmatrix} \overline{\gamma}(A - C) & 0 \\ 0 & BC^{-1}B^T \end{pmatrix} \tag{26}$$

with preconditioner Schur complement $BC^{-1}B^T$ and $\overline{\gamma} \geq \lambda_{\max}(C^{-1}A)$. In both cases, the Schur complement $BA^{-1}B^T$ or preconditioner Schur complement $BC^{-1}B^T$ can be replaced by the Schur complement preconditioner C_C^{-1}. If we use the scaling of (26) we obtain the final preconditioner

$$\mathcal{C} = \begin{pmatrix} \overline{\gamma}(A - C) & 0 \\ 0 & C_C^{-1} \end{pmatrix}. \tag{27}$$

With the notation $\mathcal{U} = (u,p)$ and $\mathcal{K} = V \times \Lambda$ we can apply the approximative projection algorithm to the Augmented Lagrangian functional (24)

$$\mathcal{U}^{k+1} = \tilde{P}_{\mathcal{K}}^{\mathcal{C}}\left(\mathcal{U}^k + \tau\mathcal{C}^{-1}(\mathcal{F} - \mathcal{A}\mathcal{U}^k)\right).$$

Now, the (approximative) projection involves only the smaller number of dual unknowns. It can be implemented by n steps of the projection method with matrix C_C^{-1} and inner product I_C.

Summing up, we developed the following algorithm. Now, we allow preconditioners C with general scaling

$$\underline{\gamma}C < A \leq \overline{\gamma}C.$$

Algorithm 3 (Neumann Projection Method)

Choose arbitrary $u^1 \in V$, $p^1 \in \Lambda$.
For $k = 1, 2, \ldots$ *do*

$$w = C^{-1}(f - Au^k - B^T p^k)$$
$$d = C_C^{-1}p^k + \tau(Bw - \underline{\gamma}(g - Bu^k))$$

For $j = 0, \ldots, n-1$ *do*

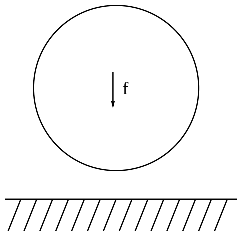
$$p^{k+\frac{j+1}{n}} = P_\Lambda^{I_C}\left(p^{k+\frac{j}{n}} + \tau_i(d - C_C^{-1}p^{k+\frac{j}{n}})\right)$$
$$u^{k+1} = u^k + \tau\overline{\gamma}^{-1}w$$

Theorem 3
Algorithm 2 has optimal time complexity on uniformly refined hierarchical meshes. This means, the time complexity for reducing the iteration error by a factor of ε, is $O(N|\log \varepsilon|)$ independent of the number of levels.

Proof: Similar to proof of Theorem 2.

6 Numerical Results

We have applied the algorithm based on Neumann domain decomposition for the solution of the following problem of Signorini. A sphere is pressed against a plane:



$$
\begin{aligned}
&\Omega = C(0,0,0;0.5) \\
&\Gamma_C = \partial\Omega \\
&E = 1, \quad \nu = 0.2 \\
&f = (0,0,-1E-3)
\end{aligned}
$$

The problem of the rigid body motions was solved by adding the small regularisation term $\varepsilon\,(u,v)_{L_2}$ with $\varepsilon = 10^{-3}$ to the bilinear form.

We used a V-cycle multigrid preconditioner C with 3 Gauss-Seidel pre-smoothing steps and 3 backward Gauss-Seidel post-smoothing steps. Due to the non-nested boundary approximations, the convergence rate of standard multigrid would depend on the small parameter ε. To overcome this difficulty, we used special grid transfer operations preserving linear functions, and therefore mapping coarse grid rigid body motions to fine grid rigid body motions. As boundary preconditioner we used the BPX preconditioner. The projection was implemented by the conjugate gradient like inner iteration of [6].

The initial mesh was constructed from a cube. Each face was split into two triangles, and each triangle was connected by a tetrahedron to the center. The following loop was performed.

- Solve the CMP up to an relative error of 10^{-4} using the Neumann DD approach.
- Apply the residual error estimator [26] for the linearized problem with fixed contact nodes. Mark all elements with element error more then 10 % of the maximal error.
- Do adaptive mesh refinement. All marked elements will be refined. In addition the red and green closures are formed.
- Prolongate the old solutions to the next mesh, where it will be used as initial guess. Displacement variables are prolongated naturally, the dual variable p, the contact stress, is prolongated by injection.

We used the finite element code FEPP and one R10000/195 MHz processor of an SGI Origin 2000 machine. In the table below we give for each level of refinement the total number of nodes (N), the number of nodes at the contact boundary (N_C), which is equal to the number of inequalities, the number of outer iterations (its) needed to reduce the error by a factor of 10^{-4}, the average number of inner iteration (av. n_i), the total time spent in the solver T_{solve} at each level, and the time spent in the projection T_{proj}.

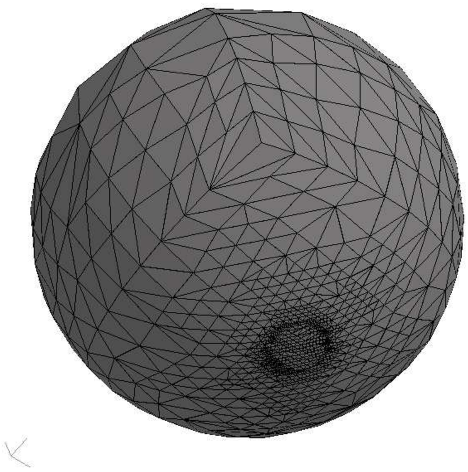
level	N	N_C	its	av. n_i	T_{solve}	T_{proj}
1	9	8	11	12.6	0.0	0.0
2	35	26	7	3.6	0.0	0.0
3	87	48	8	1.6	0.1	0.1
4	129	65	14	27.8	0.3	0.2
5	395	136	14	37.6	0.9	0.4
6	1113	286	16	78.1	3.2	1.6
7	3219	653	15	103.0	9.4	3.9
8	11074	1640	18	106.4	57.0	13.2
9	29234	3276	19	120.2	178.2	36.7
10	59354	5157	19	121.2	434.1	61.2
11	157805	10003	21	138.0	1385.6	174.1
12	305404	17950	20	165.7	2812.5	356.2

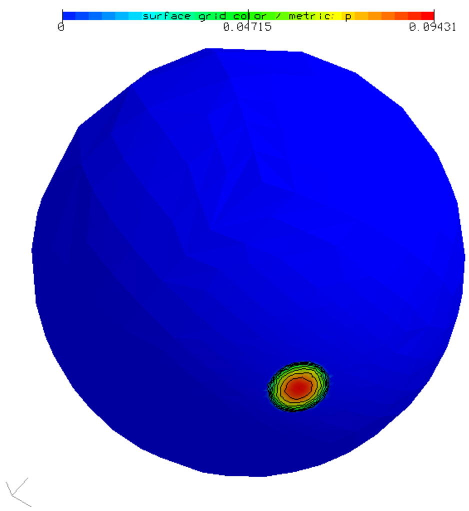
According to the analysis, the number of outer iterations is bounded. The number of inner iterations increases, but the time spent for the projection is much below the total time, which is spent mainly for the preconditioning operation $C^{-1} \times v$. Certainly due to leaving the cache, the needed time per node increases a little.

In addition to linear elements, we also used 10-node tetrahedra with quadratic shape functions. The results are given in the table below.

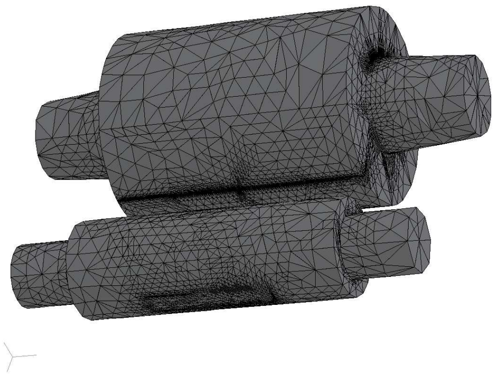
level	N	N_C	its	av. n_i	T_{solve}	T_{proj}
1	9	8	11	12.6	0.0	0.0
2	35	26	8	3.9	0.0	0.0
3	138	70	12	20.5	0.3	0.1
4	416	150	14	33.4	1.1	0.4
5	833	254	15	53.8	2.7	1.0
6	2726	598	22	82.9	14.2	4.7
7	6591	1158	19	117.8	42.2	11.4
8	18346	2422	19	177.3	134.7	35.3
9	39772	4630	20	180.5	340.5	79.2
10	136446	12870	24	248.8	1887.1	383.9

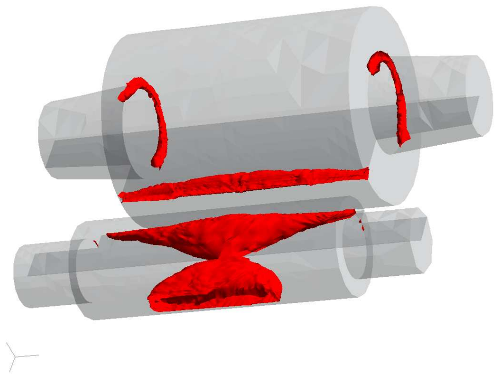
The pictures below show the mesh of 10-node tetrahedra at level 8 and the according contact stress:





As mentioned above, the mixed form and the derived algorithm are also well suited for body-body contact problems. Preliminary results are available by B. Hackl and W. Hinterberger, more details can be found in [12]. Two rolls are in contact along a small strip. The smaller roll is loaded by given surface traction, and it is asked for its vertical displacement. The adaptive mesh with 46611 Nodes and the isosurface of the von-Mises stress is drawn below:





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