Multigrid Methods for a Parameter Dependent Problem in Primal Variables

Joachim Schöberl*
joachim@numa.uni-linz.ac.at

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Abstract

In this paper we consider multigrid methods for the parameter dependent problem of nearly incompressible materials. We construct and analyze multilevel-projection algorithms, which can be applied to the mixed as well as to the equivalent, non-conforming finite element scheme in primal variables. For proper norms, we prove that the smoothing property and the approximation property hold with constants that are independent of the small parameter. Thus we obtain robust and optimal convergence rates for the W-cycle and the variable V-cycle multigrid methods. The numerical results pretty well conform the robustness and optimality of the multigrid methods proposed.

1 Introduction

We consider the linear elasticity problem to find $u \in [H_0^1(\Omega)]^2$ such that

$$2\mu \int_\Omega e(u) : e(v)\,\mathrm{dx} + \lambda \int_\Omega \operatorname{div} u \operatorname{div} v\,\mathrm{dx} = \int_\Omega \tilde{f}^T v\,\mathrm{dx}, \tag{1}$$

with the positive constants λ and μ of Lamé, the strain operator $e(u) := 0.5(\nabla u + (\nabla u)^T)$ and the volume force $\tilde{f} \in [L_2(\Omega)]^2$. We are interested in the nearly incompressible case, i.e. the Poisson ration ν is close to 0.5. Then the 'bad parameter' $\varepsilon := 2\mu/\lambda$ becomes small.

For conforming low order finite element methods the parameter ε enters disadvantageously into the discretization error estimate. This effect is also verified numerically and is well known as 'locking effect', [2]. Various non-conforming discretization methods lead to discretization errors robust for $\nu \to 0.5$, see [12], [10]. We use the mixed formulation for u and $\operatorname{div} u$ to obtain a stable saddle-point system. By the choice of non-continuous finite elements for the dual variable, it can be eliminated at element level, and we return to a symmetric positive definite finite element method.

Also the convergence rate of standard multigrid methods applied to solve the positive definite linear system deteriorates as $\nu \to 0.5$. To overcome this difficulty, robust multigrid methods have been designed for the equivalent mixed finite element scheme with penalty

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term in [21], [15], [3], [9], [4]. Related multigrid methods for the Stokes problem are analyzed in [8], [20]. The papers mainly differ in the kind of smoothing iteration used for the indefinite system.

In [18], a new multigrid method for parameter dependent problems in primal variables has been suggested and the analysis for the two-level method was given. The key components are an overlapping block-smoother capturing the divergence free basis functions, and a grid transfer operator prolongating coarse grid divergence free functions to fine grid divergence free functions. In this paper we establish the approximation and the smoothing property necessary for the multigrid analysis [14], [6].

During the analysis we switch between both equivalent algorithms, the primal one and the mixed one.

The outline of the paper is as follows. In Section 2 some available results are collected. The algorithmic aspects of the multigrid method are formulated in Section 3, the analysis is started in Section 4. Approximation property and smoothing property are proven in Section 5 and in Section 6, respectively. Numerical results are given in Section 7.

2 Stability and Discretization

We introduce the dual variable

$$p := \varepsilon^{-1} \operatorname{div} u$$

and obtain the equivalent mixed problem to find $(u, p) \in X := V \times Q := [H_0^1(\Omega)]^2 \times L_2/\mathbf{R}$ such that

$$B\left((u,p),(v,q)\right) = (f,v)_0 \qquad \forall\, (v,q) \in X, \tag{2}$$

with $f = (2\mu)^{-1}\tilde{f}$ and the bilinear-form

$$B\left((u,p),(v,q)\right) = (e(u),e(v))_0 + (\operatorname{div} u, q)_0 + (\operatorname{div} v, p)_0 - \varepsilon\,(p,q)_0, \tag{3}$$

where $(.,.)_0$ denotes the inner product in L_2 of scalar, vector valued or tensor valued functions. Clearly, B is continuous on $X \times X$ with the product norm $\|(u,p)\|_X = (\|u\|_1^2 + \|p\|_0^2)^{1/2}$. The proper stability criterion on some subspace $X_* := V_* \times Q_* \subseteq X$ is the condition

$$\sup_{(v,q)\in X_*} \frac{B((u,p),(v,q))}{\|(v,q)\|_X} \geq c\,\|(u,p)\|_X \qquad \forall\,(u,p) \in X_*. \tag{4}$$

Here and throughout the paper c will be a generic constant which is independent of the parameter ε and the mesh-size defined below and which may be different in different equations. It follows from the second inequality of Korn, the LBB condition of the Stokes problem and further estimates due to the penalty term that B is stable on $X_* = X$, see [11], [1]. We assume that Ω is a convex polygonal domain and get from [10] the regularity theorem

$$\|u\|_2 + \|p\|_1 \leq c\,\|f\|_0. \tag{5}$$

For finite element discretization we choose the subspace $X_L = V_L \times Q_L \subset X$, where V_L consists of continuous, piecewise quadratic functions, and Q_L of piecewise constant functions on a triangular mesh with mesh-size parameter h_L. The integer L defines the number of multigrid levels. We get the finite dimensional problem find $(u_L, p_L) \in X_L$ such that

$$B((u_L,p_L),(v_L,q_L)) = (f,v_L)_0 \qquad \forall\,(v_L,q_L) \in X_L. \tag{6}$$

The LBB condition is fulfilled for the pair of spaces V_L and Q_L, see [11] p. 211, which implies the stability condition (4) on $X_* = X_L$.

The essential fact is the non-continuity of the functions in Q_L leading to an easily invertible matrix for the L_2 inner product. The dual variable p_L can be eliminated element-wise and the problem can be reduced to the non-conforming symmetric and positive definite problem find $u_L \in V_L$ such that

$$A_L(u_L, v_L) = (f, v_L)_0 \qquad \forall\, v_L \in V_L, \tag{7}$$

with the bilinear-form

$$A_L(u, v) = (\varepsilon(u), \varepsilon(v))_0 + \varepsilon^{-1}(I_L^Q \operatorname{div} u, \operatorname{div} v)_0. \tag{8}$$

The operator I_L^Q denotes the L_2-orthogonal projection onto Q_L. We mention that this projection can be implemented in the element matrix assembling subroutine. Due to the equivalence of the primal and the mixed finite element method we get bounds for the discretization error that are independent of ε also for the primal version.

The author is aware of the sub-optimal convergence rate $O(h)$ for the $P_2 - P_0$ element pairing. There exist several elements with non-continuous pressure and optimal convergence rate, see [13], [11]. The element is chosen for reasons of simpler notation and implementation, but the following analysis is not limited to the special element.

3 The Multigrid Algorithm

For the application of multigrid solvers a sequence of uniformly refined triangulations $\mathcal{T}_l$ of mesh-size h_l and the corresponding nested $P_2 - P_0$ finite element spaces

$$X_1 = V_1 \times Q_1 \subset X_2 = V_2 \times Q_2 \subset \ldots \subset X_L = V_L \times Q_L \tag{9}$$

are used. By means of the computable Q-orthogonal projection operators $I_l^Q : Q \to Q_l$ we define the bilinear-forms

$$A_l(u, v) = (e(u), e(v))_0 + \varepsilon^{-1}\,(I_l^Q \operatorname{div} u, \operatorname{div} v)_0 \quad \forall\, u, v \in V, \tag{10}$$

for $l = 1, \ldots, L$. We mention that the forms are defined on the infinite dimensional space V. We define norms $\|u\|_{A_l} := A_l(u, u)^{1/2}$ and the L_2 self-adjoint operators $A_l : V_l \to V_l$ as $(A_l u_l, v_l)_0 = A_l(u_l, v_l)$, $\forall\, u_l, v_l \in V_l, l = 1, \ldots, L$. It is clear that $A_l(.,.)$ estimates $A_{l+1}(.,.)$ from below, i.e.

$$A_{l+1}(u, u) \geq A_l(u, u) \qquad \forall\, u \in V,$$

but the converse estimate does not hold with some constant independently bounded in ε.

This fact requires special grid transfer operators, which are constructed as follows. On each level $l = 2, \ldots, L$ we define the subspace of functions which vanish on the boundaries of the coarse grid elements

$$V_{l,T} := \prod_{T \in \mathcal{T}_{l-1}} [H_0^1(T)]^2 \cap V_l. \tag{11}$$

It splits orthogonally into $|\mathcal{T}_{l-1}|$ subspaces. Each of them is generated by the basis functions belonging to the nodes inside a triangle of the coarser grid, see Figure 1.

We define the projection operator $P_{l,T}^{A_l} : V \to V_{l,T}$ such that

$$A_l(P_{l,T}^{A_l} u, v_{l,T}) = A_l(u, v_{l,T}) \qquad \forall\, u \in V,\ \forall\, v_{l,T} \in V_{l,T}. \tag{12}$$

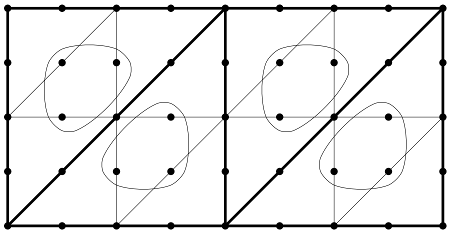


Figure 1: Subspace $V_{l,T}$ used in prolongation

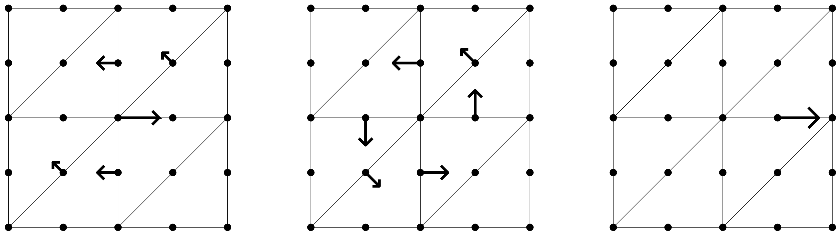


Figure 2: Basis functions for kernel of $I_l^Q \operatorname{div}$

The co-projection $I - P_{l,T}^{A_l}$ is the discrete harmonic extension on each coarse grid triangle. It can be computed fast and will be used as prolongation operator. We use the natural embedding $V_{l-1} \subset V_l$ without denoting it by any symbol.

$$\boxed{\begin{array}{l}\text{Prolongation } u_{l-1} \to u_l\text{:} \\ \qquad\qquad u_l = (I - P_{l,T}^{A_l}) u_{l-1}\end{array}} \tag{13}$$

The idea of this prolongation is to lift coarse grid div-free functions to fine grid div-free functions. As we will see later, this prolongation is continuous in the sense of

$$\|(I - P_{l,T}^{A_l})\, u_{l-1}\|_{A_l} \leq c\, \|u_{l-1}\|_{A_{l-1}} \qquad \forall\, u_{l-1} \in V_{l-1}.$$

We define the operator $E_l^{-1} : V_l \to V_l$ as $E_l^{-1} = P_{l,T}^{A_l} A_l^{-1}$ such that the prolongation can be rewritten as

$$I - P_{l,T}^{A_l} = I - E_l^{-1} A_l.$$

The operator E_l is self-adjoint with respect to $(.,.)_0$. In matrix form E_l is the restriction of A_l to the degrees of freedom spanning the space $V_{l,T}$. Using the L_2-orthogonal projection $P_{l-1}^{L_2} : V \to V_{l-1}$, the $(.,.)_0$ - adjoint restriction operator is $P_{l-1}^{L_2}(I - A_l E_l^{-1})$. We mention that the projection $P_{l-1}^{L_2}$ is required for notation only, it does not enter into the computation.

Also the smoother must be properly designed. A damped Richardson smoother $(I - \tau A_l)$ would need a damping parameter τ proportional to ε. Thus the components of the error in the kernel of A_l would be smoothed out very slow, as ε becomes small. The suggested smoother is a block Jacobi smoother, which takes care of the kernel of $I_l^Q \operatorname{div}$. On a simply connected domain with only one part of natural boundary conditions, the kernel of $I_l^Q \operatorname{div}$ is spanned by basis functions drawn in Figure 2, similar to [11], pp 268.

These kernel basis functions are captured by subspaces $V_{l,i}$ generated by nodal basis functions belonging to the nodes drawn in Figure 3. For all $l = 2, \ldots, L$, this leads to the definition of the n_l subspaces

$$V_{l,i} = [H_0^1(\Omega_i)]^2 \cap V_l \tag{14}$$

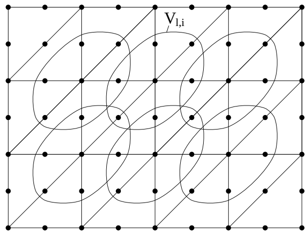


Figure 3: Subspaces containing div-free basis functions

assigned to the corner nodes $N_{l,i}, i = 1, \ldots, n_l$, of the triangulation $\mathcal{T}_l$. Here, Ω_i is the closure of the union over elements in $\mathcal{T}_l$ adjacent to the node $N_{l,i}$. We define the projections $P_{l,i}^{A_l} : V \to V_{l,i}$ as

$$A_l(P_{l,i}^{A_l} u, v_{l,i}) = A_l(u, v_{l,i}) \qquad \forall\, u \in V \;\; \forall\, v_{l,i} \in V_{l,i}. \tag{15}$$

By means of these subspaces we define the block Jacobi smoother as

$$\boxed{\begin{array}{l}\text{Smoother } S_l^A : V_l \to V_l\text{:} \\ \qquad\qquad S_l^A = I - \tau \sum_{i=1}^{n_l} P_{l,i}^{A_l}.\end{array}} \tag{16}$$

The necessary damping parameter depends only on the number of overlapping spaces, which is bounded for shape regular elements. Especially, it does not depend on ε and l. We assume that τ is small enough to ensure only positive eigenvalues of S_l^A. In the numerical examples, we will also use the according block Gauss-Seidel smoother, which does not need any damping at all. We also define the operator $D_l^{-1} : V_l \to V_l$ as

$$D_l^{-1} = \sum_{i=1}^{n_l} P_{l,i}^{A_l} A_l^{-1}, \tag{17}$$

which corresponds in matrix form to the sum over local inverses. By means of D_l the smoother can be written as

$$S_l^A = I - \tau D_l^{-1} A_l. \tag{18}$$

Now we can state the multigrid algorithm in recursive form. We apply m_l smoothing steps on the level l and perform either the V-cycle ($q = 1$) or the W-cycle ($q = 2$).

Algorithm 1
Procedure $MG(u, f, l)$
if $l = 1$

$$MG(u, f, l) = A_1^{-1} f$$

else

$$\begin{aligned}
& u^{1,0} = u \\
& do\ j = 1, \ldots, m_l \\
& \qquad u^{1,j} = u^{1,j-1} + \tau D_l^{-1}(f - A_l u^{1,j-1}) \\
& d^1 = f - A_l u^{1,m_l} \\
& d^2 = P_{l-1}^{L_2}(I - A_l E_l^{-1}) d^1 \\
& u^{2,0} = 0
\end{aligned}$$

$do\ j = 1, \ldots, q$
$\quad u^{2,j} = MG(u^{2,j-1}, d^2, l-1)$
$u^3 = (I - E_l^{-1} A_l) u^{2,q}$
$u^{4,0} = u^{1,m_l} + u^3$
$do\ j = 1, \ldots, m_l$
$\quad u^{4,j} = u^{4,j-1} + \tau D_l^{-1}(f - A_l u^{4,j-1})$
$MG(u, f, l) = u^{4,m_l}$

4 The Multigrid Analysis

In this section we start the analysis of the multigrid method. First, we specify a multigrid method for the mixed form and prove equivalence to the algorithm stated in the last chapter. We define an L_2-like norm, for which we will prove the approximation property and the smoothing property in the following two chapters.

For $l = 1, \ldots, L$ we define the subspaces

$$X_{l,0} := \left\{ (u_l, p_l) \in X_l : I_l^Q \operatorname{div} u_l = \varepsilon\, p_l \right\}. \tag{19}$$

We will use the relation

$$B((u_l, p_l), (0, q_l)) = 0 \qquad \forall\, (u_l, p_l) \in X_{l,0}, \ \ \forall\, q_l \in Q_l \tag{20}$$

later. From (10) and (19) it follows that

$$B((u_l, p_l), (v_l, 0)) = A_l(u_l, v_l) \qquad \forall\, (u_l, p_l) \in X_{l,0}, \ \ \forall\, v_l \in V_l \tag{21}$$

is valid. We extend definition (14) to local mixed spaces

$$X_{l,i} := V_{l,i} \times Q_{l,i} \qquad \text{with} \qquad Q_{l,i} = (L_2(\Omega_{l,i})/\mathbf{R}) \cap Q_l \tag{22}$$

for $i = 1, \ldots, n_l$, $l = 1, \ldots, L$. We also extend the spaces $V_{l,T}$ of the prolongation defined in (11) to the mixed spaces

$$X_{l,T} := V_{l,T} \times Q_{l,T} \qquad \text{with} \qquad Q_{l,T} := \prod_{T \in \mathcal{T}_{l-1}} (L_2(T)/\mathbf{R}) \cap Q_l, \tag{23}$$

$l = 2, \ldots, L$. The spaces are designed such that the orthogonal decomposition

$$Q_l = Q_{l-1} \oplus Q_{l,T} \tag{24}$$

and

$$B((u_T, p_T), (0, q_{l-1})) = 0 \qquad \forall\, (u_T, p_T) \in X_{l,T}, \ \ \forall\, q_{l-1} \in Q_{l-1} \tag{25}$$

hold. In addition to the norm $\|.\|_X$ we define the energy-norm

$$\|(u,p)\|_B := B((u,p),(u,-p))^{1/2} = (\|e(v)\|_0^2 + \varepsilon\, \|p\|_0^2)^{1/2},$$

which by Korn's inequality is equivalent to the norm $(\|u\|_1^2 + \varepsilon \|p\|_0^2)^{1/2}$.

For any subspace $X_* \subset X$ the projection $P_*^B : X \to X_*$ is defined by

$$B(P_*^B(u,p),(v,q)) = B((u,p),(v,q)) \qquad \forall\, (v,q) \in X_*. \tag{26}$$

We will use projections to X_{l-1}, $X_{l,T}$ and $X_{l,i}$. The next lemma collects some of their properties.

Lemma 1

1. *All subspaces X_l, $X_{l,i}$ and $X_{l,T}$ fulfill the stability condition (4) with one common constant c.*

2. *The projections P^B_{l-1}, $P^B_{l,i}$, $P^B_{l,T}$ are well defined and uniformly bounded on X_l with respect to the $\|.\|_X$-norm.*

3. *P^B_{l-1} maps $X_{l,0}$ into $X_{l-1,0}$.*

4. *$P^B_{l,i}$ and $P^B_{l,T}$ map $X_{l,0}$ into itself, and they are bounded by 1 with respect to the $\|.\|_B$-norm on $X_{l,0}$.*

5. *The co-projection $I - P^B_{l,T}$ maps $X_{l-1,0}$ into $X_{l,0}$.*

Proof: 1. Stability condition for the spaces are standard. The spaces $V_{l,i}$ and the factor-spaces of $V_{l,T}$ can be derived from a finite number of spaces by scaling and translation, and these transformations do not change the stability constant. Thus the common constant is the maximum of a finite number of stability constants.
2. The continuity of P^B_* with respect to $\|.\|_X$-norm follows from the stability condition (4)

$$\|P^B_*(u,p)\|_X \le c \sup_{(v,q)\in X_*} \frac{B(P^B_*(u,p),(v,q))}{\|(v,q)\|_X} = c \sup_{(v,q)\in X_*} \frac{B((u,p),(v,q))}{\|(v,q)\|_X} \le c\,\|(u,p)\|_X .$$

3. For $(u,p) \in X_{l,0}$ we get

$$B(P^B_{l-1}(u,p),(0,q_{l-1})) = B((u,p),(0,q_{l-1})) = 0 \qquad \forall\, q_{l-1} \in Q_{l-1}.$$

4. Next set $(\hat{u},\hat{p}) = P^B_{l,i}(u,p)$. We now decompose a function $q \in Q_l$ orthogonally into $q_1 \in Q_{l,i}$ and $q_2 = (q,1)_{0,\Omega_{l,i}}/|\Omega_{l,i}|$ in $\Omega_{l,i}$ and $q_2 = q$ in $\Omega \setminus \Omega_{l,i}$. $B((\hat{u},\hat{p}),(0,q_1))$ vanishes by definition of the projection, $B((\hat{u},\hat{p}),(0,q_2)) = (\operatorname{div}\hat{u},q_2)_{0,\Omega_{l,i}} - \varepsilon\,(\hat{p},q_2)_{0,\Omega_{l,i}} = 0$ is achieved by Green's theorem and definition of $Q_{l,i}$. By (24), (25) and the same arguments $P^B_{l,T} \in X_{l,0}$ is proven.

Now, let $(u,p) \in X_{l,0}$ and X_* such that $(\hat{u},\hat{p}) = P^B_*(u,p) \in X_{l,0}$. Then,

$$\begin{aligned}
&\|(\hat{u},\hat{p})\|^2_B = B((\hat{u},\hat{p}),(\hat{u},-\hat{p})) = B((u,p),(\hat{u},-\hat{p})) = \\
&= \quad B((u,p),(\hat{u},-\hat{p})) + B((u,p),(0,\hat{p})) - B((0,p),(\hat{u},\hat{p})) = (e(u),e(\hat{u}))_0 + \varepsilon\,(p,\hat{p})_0 \\
&\le \quad (\|e(u)\|^2_0 + \varepsilon\,\|p\|^2_0)^{1/2}(\|e(\hat{u})\|^2_0 + \varepsilon\,\|\hat{p}\|^2_0)^{1/2} = \|(u,p)\|_B\,\|(\hat{u},\hat{p})\|_B
\end{aligned}$$

gives the upper bound 1.

5. Let $(u,p) \in X_{l-1,0}$. Then

$$B((I - P^B_{l,T})(u,p),(0,q_{l-1})) = 0 \qquad \forall\, q_{l-1} \in Q_{l-1}$$

holds because of the assumption and (25). The same form tested with $q \in Q_{l,T}$ vanishes because of the definition of the projection, and Q_l can be decomposed by (24).

□

We also define projections $P^{A_l,A_k}_* : V_k \to V_* \subset V_l$ by

$$A_l(P^{A_l,A_k}_* u, v) = A_k(u,v) \qquad \forall v \in V_* \tag{27}$$

and set $P^{A_l}_* = P^{A_l,A_l}_*$. This definition is consistent with (12) and (15).

Lemma 2
Let $X_* = V_* \times Q_* \subset X_l$ and $(\hat{u},\hat{p}) = P^B_*(u,p)$.

1. *If $(u,p) \in X_{k,0}$ and $(\hat u,\hat p) \in X_{l,0}$, then $\hat u = P_*^{A_l,A_k} u$.*

2. *If $(u,p) - (\hat u,\hat p) \in X_{l,0}$, then $\hat u = P_*^{A_l} u$.*

Proof: To verify statement 1 we use (21) and obtain

$$A_l(\hat u, v) = B((\hat u,\hat p),(v,0)) = B((u,p),(v,0)) = A_k(u,v) \qquad \forall\, v \in V_*.$$

Statement 2 is checked by

$$A_l(\hat u - u, v) = B((\hat u - u, \hat p - p),(v,0)) = 0 \qquad \forall v \in V_*.$$

□

Algorithm 1 leads to the multigrid operator M_L^A, which for $l = 2,\dots,L$ fulfills the recursion

$$\begin{aligned} M_1^A &= 0,\\ M_l^A &= (S_l^A)^{m_l}\left(I - (I - P_{l,T}^{A_l})(I - (M_{l-1}^A)^q)P_{l-1}^{A_{l-1},A_l}(I - P_{l,T}^{A_l})\right)(S_l^A)^{m_l}. \end{aligned} \tag{28}$$

We define the corresponding multigrid operator for the mixed system as

$$\begin{aligned} M_1^B &= 0,\\ M_l^B &= (S_l^B)^{m_l}\left(I - (I - P_{l,T}^{B})(I - (M_{l-1}^B)^q)P_{l-1}^{B}(I - P_{l,T}^{B})\right)(S_l^B)^{m_l}, \end{aligned}$$

with the smoothing operators

$$S_l^B := I - \tau \sum_{i=1}^{n_l} P_{l,i}^B$$

for $2 \le l \le L$. The iteration for the mixed system is well defined for the limit case $\varepsilon = 0$, too.

Theorem 1 (Equivalence of Algorithms)
Both multigrid procedures are equivalent, namely for $(u_l,p_l) \in X_{l,0}$ there holds

$$(\hat u_l, \hat p_l) := M_l^B(u_l,p_l) \qquad \textit{fulfills} \qquad \hat u_l = M_l^A(u_l). \tag{29}$$

Proof: By induction on l, Lemma 1 and Lemma 2.

□

In the following chapters we will need the approximation properties of the finite element spaces. Let I_l^V be the Lagrange interpolator into V_l. Recall the (local) L_2 projector I_l^Q to Q_l and define the product operator $I_l^X = \left(I_l^V, I_l^Q\right) : X \to X_l$. Then the approximation inequalities

$$\|u - I_l^V u\|_1 \le c\, h_l \|u\|_2 \qquad \text{and} \qquad \|p - I_l^Q p\|_0 \le c\, h_l \|p\|_1 \tag{30}$$

hold.

We define the norm

$$\|(u,p)\|_{l,0}^2 := h_l^{-2}\|u\|_0^2 + \varepsilon\|p\|_0^2 + \|I_{l-1}^Q p\|_0^2. \tag{31}$$

On the space $X_{l,0}$, it is identical to the norm

$$\|u\|_{l,0}^2 := h_l^{-2}\|u\|_0^2 + \varepsilon^{-1}\|I_l^Q \operatorname{div} u\|_0^2 + \varepsilon^{-2}\|I_{l-1}^Q \operatorname{div} u\|_0^2 \tag{32}$$

on V_l. These norms will be used in the multigrid proof for measuring smoothness.

The main theorem of this paper is

Theorem 2 (Two-Grid Convergence)
The two-grid operator $\hat{M}_l^A$ can be estimated by

$$\|\hat{M}_l^A\|_{A_l} \leq c\, m_l^{-1/4}, \tag{33}$$

with a constant c independent of l and ε. The two-grid operator $\hat{M}_l^B$ maps $X_{l,0}$ into itself and is bounded on $X_{l,0}$ by

$$\|\hat{M}_l^B\|_B \leq c\, m_l^{-1/4} \tag{34}$$

with a constant c independent of l and ε.

Proof: Define for $(u^0, p^0) \in X_{l,0}$

$$(u^1, p^1) = \left(I - (I - P_{l,T}^B) P_{l-1}^B (I - P_{l,T}^B)\right)(u^0, p^0) \tag{35}$$

and

$$(u^2, p^2) = (S_l^B)^{m_l}(u^1, p^1). \tag{36}$$

By the previous lemmata we get $(u^1, p^1) \in X_{l,0}$ and

$$u^1 = \left(I - (I - P_{l,T}^{A_l}) P_{l-1}^{A_{l-1},A_l} (I - P_{l,T}^{A_l})\right) u^0.$$

In Section 5 we will prove the approximation property (see Theorem 4)

$$\|(u^1, p^1)\|_{l,0} = \|u^1\|_{l,0} \leq c\,\|u^0\|_{A_l} = c\,\|(u^0, p^0)\|_B \tag{37}$$

using the mixed form. We also get $(u^2, p^2) \in X_{l,0}$ and

$$u^2 = (S_l^A)^{m_l} u^1.$$

In Section 6 we will prove the smoothing property (see Theorem 5)

$$\|(u^2, p^2)\|_B = \|u^2\|_{A_l} \leq c\, m^{-1/4} \|u^1\|_{l,0} = c\, m^{-1/4} \|(u^1, p^1)\|_{l,0} \tag{38}$$

using the primal form. Combining both properties proves the theorem.

□

The following theorem follows by standard techniques [14], [6], [5].

Theorem 3 (Multigrid Convergence)

- *The norm of the W-cycle operator is bounded independently of L and ε if the number of smoothing steps m_l is sufficiently large.*
- *The variable V-cycle operator with $m^l = 2^{L-l}$ leads to a preconditioner $C_L^{-1} := (I - M_L^A) A_L^{-1}$ with condition number $\kappa(C_L^{-1} A_L)$ bounded independently of L and ε.*

5 Approximation Property

The coarse grid operator $(u_1, p_1) \in X_{l,0} \to (u_5, p_5) \in X_{l,0}$ is split into

$$\begin{aligned} (u_2, p_2) &= (I - P_{l,T}^B)(u_1, p_1), \\ (u_3, p_3) &= P_{l-1}^B (u_2, p_2), \\ (u_4, p_4) &= (I - P_{l,T}^B)(u_3, p_3), \\ (u_5, p_5) &= (u_1, p_1) - (u_4, p_4). \end{aligned} \tag{39}$$

Theorem 4 (Approximation Property)
Let $(u_1, p_1) \in X_{l,0}$ *and compute* (u_5, p_5) *by (39). Then the approximation property*

$$\|(u_5, p_5)\|_{l,0} \le c\,\|(u_1, p_1)\|_B \tag{40}$$

is valid.

Proof: We use the triangle inequality and the three lemmata proven below to obtain the result

$$\begin{aligned}\|(u_5, p_5)\|_{l,0} &= \|(u_1, p_1) - (u_4, p_4)\|_0 \\ &\le \|(u_1, p_1) - (u_2, p_2)\|_{l,0} + \|(u_2, p_2) - (u_3, p_3)\|_{l,0} + \|(u_3, p_3) - (u_4, p_4)\|_{l,0} \\ &\le c\,\|(u_1, p_1)\|_B.\end{aligned}$$

□

Lemma 3
With the notation of (39) there holds

$$\|(u_2, p_2)\|_B + \|(u_2, p_2) - (u_1, p_1)\|_{l,0} + \|p_2 - I^Q_{l-1} p_2\|_0 \le c\,\|(u_1, p_1)\|_B. \tag{41}$$

Proof: Lemma 1 gives $\|P^B_{l,T}(u_1, p_1)\|_B \le \|(u_1, p_1)\|_B$, which bounds the first term. The second term is bounded due to the norm equivalence $\|.\|_B \sim \|.\|_{l,0}$ on $X_{l,T}$. From $p_2 - I^Q_{l-1} p_2 \in Q_{l,T}$, stability (4) of $X_{l,T}$, orthogonality (25) and the definition of $P^B_{l,T}$ we obtain

$$\begin{aligned}\|p_2 - I^Q_{l-1} p_2\|_0 &\le c \sup_{(v,q)\in Q_{l,T}} \frac{B((0, p_2 - I^Q_{l-1} p_2), (v, q))}{\|(v, q)\|_X} \\ &= c \sup_{(v,q)\in Q_{l,T}} \frac{B((0, p_2), (v, q))}{\|(v, q)\|_X} = c \sup_{(v,q)\in Q_{l,T}} \frac{B((-u_2, 0), (v, q))}{\|(v, q)\|_X} \le c\,\|u_2\|_1.\end{aligned}$$

□

Lemma 4
With the notation of (39) there holds

$$\|(u_3, p_3)\|_B + \|(u_3, p_3) - (u_2, p_2)\|_{l,0} \le c\,\|(u_1, p_1)\|_B. \tag{42}$$

Proof: By stability of X_{l-1}, the definition of (u_3, p_3), continuity of $B(.,.)$ and Lemma 3 we get

$$\begin{aligned}\|u_3\|_1 + \|p_3 - I^Q_{l-1} p_2\|_0 &\le c \sup_{(v,q)\in Q_{l-1}} \frac{B((u_3, p_3 - I^Q_{l-1} p_2), (v, q))}{\|(v, q)\|_X} \\ &= c \sup_{(v,q)\in Q_{l-1}} \frac{B((u_2, p_2 - I^Q_{l-1} p_2), (v, q))}{\|(v, q)\|_X} \\ &\le c\,(\|u_2\|_1 + \|p_2 - I^Q_{l-1} p_2\|_0) \le c\,\|(u_1, p_1)\|_B,\end{aligned}$$

and in combination with Lemma 3

$$\|p_3 - p_2\|_0 \le \|p_3 - I^Q_{l-1} p_2\|_0 + \|p_2 - I^Q_{l-1} p_2\|_0 \le c\,\|(u_1, p_1)\|_B.$$

This gives also

$$\varepsilon\|p_3\|_0^2 \le 2\varepsilon\|p_2 - p_3\|_0^2 + 2\varepsilon\|p_2\|_0^2 \le c\,\|(u_1, p_1)\|_B^2.$$

We state the dual problem on X

$$B((\varphi,\psi),(v,q)) = (u_2 - u_3, v)_0 \qquad \forall\,(v,q)\in X$$

and get the L_2 estimate by Galerkin orthogonality, approximation (30) and regularity (5)

$$\begin{aligned}
\|u_2-u_3\|_0^2 &= B((\varphi,\psi),(u_2-u_3,p_2-p_3)) \\
&= B((\varphi,\psi)-(I_{l-1}^V\varphi, I_{l-1}^Q\psi),(u_2-u_3,p_2-p_3)) \\
&\le c\,(\|\varphi - I_{l-1}^V\varphi\|_1 + \|\psi - I_{l-1}^Q\psi\|_0)\,(\|u_2-u_3\|_1 + \|p_2-p_3\|_0) \\
&\le c\,h\,(\|\varphi\|_2 + \|\psi\|_1)\,(\|u_2\|_1 + \|u_3\|_1 + \|p_2-p_3\|_0) \\
&\le c\,h\,\|u_2-u_3\|_0\,\|(u_1,p_1)\|_B.
\end{aligned}$$

Dividing by $\|u_2-u_3\|_0$ we obtain the result.

□

Lemma 5
With the notation of (39) there holds

$$\|(u_4,p_4)-(u_3,p_3)\|_{l,0} \le c\,\|(u_1,p_1)\|_B. \tag{43}$$

Proof: Friedrichs' inequality on $V_{l,T}$, stability (4), Galerkin - and orthogonality (25) give

$$\begin{aligned}
&\|(u_4,p_4)-(u_3,p_3)\|_{l,0} \le c\,\|(u_4,p_4)-(u_3,p_3)\|_X \\
&\quad\le c \sup_{(v,q)\in X_{l,T}} \frac{B((u_3-u_4,p_3-p_4),(v,q))}{\|(v,q)\|_X} = c \sup_{(v,q)\in X_{l,T}} \frac{B((u_3,p_3),(v,q))}{\|(v,q)\|_X} \\
&\quad= c \sup_{(v,q)\in X_{l,T}} \frac{B((u_3,0),(v,q))}{\|(v,q)\|_X} \le \|u_3\|_V,
\end{aligned}$$

and the proof if complete.

□

6 Smoothing Property

In this chapter we prove the smoothing property

$$\|(I-\tau D_l^{-1}A_l)^m u\|_{A_l} \le c\,m^{-1/4}\|u\|_{l,0}. \tag{44}$$

Recall that we have chosen τ such that $\|\tau D_l^{-1}A_l\|_{A_l} \le 1$. The estimate

$$\|(I-\tau D_l^{-1}A_l)^m u\|_{A_l}^2 = (D_l^{-1}A_l(I-\tau D_l^{-1}A_l)^{2m}u,u)_{D_l} \le cm^{-1}\|u\|_{D_l}^2 \tag{45}$$

is well established in multigrid theory [14].

By additive Schwarz techniques [22], [17] the induced norm $\|u\|_{D_l} = (D_l u,u)_0^{1/2}$ can be expressed by

$$\|u\|_{D_l}^2 = \inf_{\substack{u=\sum u_{l,i} \\ u_{l,i}\in V_{l,i}}} \sum \|u_{l,i}\|_{A_l}^2.$$

If the estimate $\|u\|_{D_l} \le c\,\|u\|_{l,0}$ would be true, the smoothing property would be proven. Unfortunately, it is not. The essential part of this section is the proof of the estimate

$$\|u\|_{[D_l,A_l]_{1/2}} \le c\,\|u\|_{l,0}, \tag{46}$$

where $\|.\|_{[D_l,A_l]_{1/2}}$ is the interpolation norm between $\|.\|_{D_l}$ and $\|.\|_{A_l}$ with parameter 1/2. We use the real method of interpolation of Lions and Peetre [16], see also [7]. Inequalities (45) and (46) immediately give the smoothing property

$$\begin{aligned} \|(I-\tau D_l^{-1}A_l)^m u\|_{A_l} &\le c\,\|(I-\tau D_l^{-1}A_l)^m u\|_{[A_l,A_l]_{1/2}} \\ &\le cm^{-1/4}\|u\|_{[D_l,A_l]_{1/2}} \le cm^{-1/4}\|u\|_{l,0}. \end{aligned} \tag{47}$$

We define the bilinear-form for the limit case $\varepsilon = 0$ as

$$B_0((u,p),(v,q)) = (e(u),e(v))_0 + (\operatorname{div} u, q)_0 + (\operatorname{div} v, p)_0. \tag{48}$$

To establish (46) we split $u = u_1 + u_2 + u_3$ by solving for $(u_i, p_i) \in X_l$ such that

$$\begin{aligned} B_0((u_1,p_1),(v,q)) &= B_0((u,0),(v,0)), \\ B_0((u_2,p_2),(v,q)) &= B_0((u,0),(0,q-I_{l-1}^Q q)), \\ B_0((u_3,p_3),(v,q)) &= B_0((u,0),(0,I_{l-1}^Q q)) \qquad \forall (v,q)\in X_l. \end{aligned} \tag{49}$$

The splitting is constructed such that u_1 is discrete divergence free, u_2 has non-smooth divergence and u_3 has smooth divergence.

Theorem 5 (Smoothing Property)
The estimate (46) and therefore the smoothing property (44) are valid.

Proof: We split u using (49), apply the triangle inequality, Lemma 7 - 9, and Lemma 6 below to obtain (46) by

$$\begin{aligned} \|u\|_{[D_l,A_l]_{1/2}} &\le \|u_1\|_{[D_l,A_l]_{1/2}} + \|u_2\|_{[D_l,A_l]_{1/2}} + \|u_3\|_{[D_l,A_l]_{1/2}} \\ &\le c\,(\|u_1\|_{l,0} + \|u_2\|_{l,0} + \|u_3\|_{l,0}) \\ &\le c\,\|u\|_{l,0}. \end{aligned}$$

The smoothing property (44) follows by the estimates (47).

□

Lemma 6
The decomposition (49) is stable in $\|.\|_{l,0}$ norm, namely

$$\|u_1\|_{l,0} + \|u_2\|_{l,0} + \|u_3\|_{l,0} \le c\,\|u\|_{l,0}. \tag{50}$$

Proof: By stability (4) we get the bounds $\|u_1\|_1 + \|p_1\|_0 \le c\,\|u\|_1$ and $\|u_2\|_1 + \|p_2\|_0 \le c\,\|I_l^Q \operatorname{div} u\|_0$. First, we bound $\|u_1\|_{l,0}^2 = h^{-2}\|u_1\|_0^2$. The solution of the dual problem find $(\varphi,\psi) \in X$ such that

$$B_0((\varphi,\psi),(v,q)) = (u_1, v)_0 \qquad \forall\,(v,q)\in X,$$

is bounded by $\|\varphi\|_2 + \|\psi\|_1 \le c\,\|u_1\|_0$. By Galerkin orthogonality, approximation, regularity, and the inverse inequality $h\,\|u\|_1 \le c\,\|u\|_0$ we obtain

$$\begin{aligned} &\|u_1\|_0^2 = B_0((\varphi,\psi),(u_1,p_1)) \\ &\quad = B_0((\varphi,\psi) - I_l^X(\varphi,\psi),(u_1,p_1)) + B_0(I_l^X(\varphi,\psi) - (\varphi,\psi),(u,0)) + B_0((\varphi,\psi),(u,0)) \\ &\quad \le c\,(h\,(\|\varphi\|_2 + \|\psi\|_1)\,(\|u_1\|_1 + \|p_1\|_0) + h\,(\|\varphi\|_2 + \|\psi\|_1)\,\|u\|_1 + (\|\varphi\|_2 + \|\psi\|_1)\|u\|_0) \\ &\quad \le c\,(h\,\|u_1\|_0\,\|u\|_1 + \|u_1\|_0\|u\|_0) \le c\,\|u_1\|_0\,\|u\|_0. \end{aligned}$$

Next, we estimate
$$h^{-2}\|u_2\|_0^2 \leq c\,\|I_l^Q \operatorname{div} u\|_0^2 \leq c\,\varepsilon\,\|u\|_{l,0}^2. \tag{51}$$
Therefore let
$$B_0((\varphi,\psi),(v,q)) = (u_2,v)_0 \qquad \forall\,(v,q) \in X,$$
then we get by $B((u_2,p_2),(v_l,q_{l-1})) = 0 \quad \forall\,v_l \in V_l,\ q_{l-1} \in Q_{l-1}$
$$\begin{aligned}\|u_2\|_0^2 &= B_0((\varphi,\psi) - (I_l^V\varphi, I_{l-1}^Q\psi),(u_2,p_2))\\ &\leq\ ch(\|\varphi\|_2 + \|\psi\|_1)\,(\|u_2\|_1 + \|p_2\|_0) \leq c\,\|u_2\|_0\,h\,\|I_l^Q \operatorname{div} u\|_0.\end{aligned}$$
The last term u_3 is bounded by the triangle inequality.

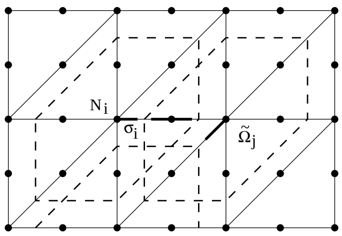
□

The discrete divergence free part u_1 is estimated by lifting to the potential space and Sobolev-Space interpolation in the next lemma.

Lemma 7
Let u_1 be defined in (49). Then the estimate
$$\|u_1\|_{[D_l,A_l]_{1/2}} \leq c\,\|u_1\|_{l,0} \tag{52}$$
is valid.

Proof: First, we define a lifting procedure $E : V_l \to V$ and a left-inverse interpolation operator $\Pi : V \to V_l$ between discrete divergence free and continuous divergence free functions. Therefore let $X^+ := V^+ \times Q^+ := \prod_T(H_0^1(T) \times L_2(T)/\mathbf{R})$ and set $Eu_1 = u_1 - w$, with $(w,p) \in X^+$ such that
$$B_0((w,p),(v,q)) = B_0((u_1,0),(0,q)) \qquad \forall\,(v,q) \in X^+. \tag{53}$$
Hence, $\operatorname{div} Eu_1 = 0$, and by stability and Friedrichs' inequality
$$\|Eu_1\|_1 + h^{-1}\|Eu_1\|_0 \leq ch^{-1}\|u_1\|_0.$$
Because Ω is assumed to be convex, there exists a potential $\varphi \in H_0^2(\Omega)$ such that
$$Eu_1 = \operatorname{rot}\varphi \qquad \|\varphi\|_2 + h^{-1}\|\varphi\|_1 \leq ch^{-1}\|u_1\|_0.$$

The interpolation is a modification of the Scott-Zhang interpolation [19]. First, shrink all Ω_i to $\widetilde{\Omega_i} = N_i + 0.9(\Omega_i - N_i)$. For each node N_i select an edge e_i and a set σ_i such that $N_i \in \sigma_i \subset e_i$, $|\sigma_i| \geq c|e_i|$ and $\sigma_i \cap \widetilde{\Omega}_j = \emptyset\ \forall$ corner nodes $N_i \neq N_j$, see the figure below. We set $\sigma = \cup\sigma_i$.



Following [19], we construct a $L_2(\sigma)$-biorthogonal basis $\{l_i \in L_2(\sigma_i)\}$ to the nodal basis $\{p_i\}$. The projection operator $\Pi_1 : V \to V_l$
$$\Pi_1 v := \sum (v,l_i)_{L_2(\sigma)} p_i$$

is well defined on H^1 and the approximation is of optimal order

$$\|v - \Pi_1\, v\|_m \leq c\, h^{1-m}\, \|v\|_1, \qquad m = 0, 1. \tag{54}$$

If $v \in V$ is such that $\operatorname{supp} v \in \tilde{\Omega}_i$, then $\Pi_1 v \in V_{l,i}$. The operator $\Pi_2 : V \to V_l$ defined by

$$\begin{aligned} \Pi_2 v(N_i) &= 0 \qquad && \forall \, \text{corner nodes } N_i \\ \int_{e_i} \Pi_2 v \, \mathrm{ds} &= \int_{e_i} v \, \mathrm{ds} \qquad && \forall \, \text{edges } e_i \end{aligned}$$

is standard for Stokes problems [11], p. 211, and fulfills

$$\|\Pi_2 v\|_1 \leq c\, (\|v\|_1 + h^{-1}\|v\|_0).$$

Then the projection operator

$$\Pi_l := \Pi_2 (I - \Pi_1) + \Pi_1$$

fulfills $\|\Pi_l\|_1 \leq c$ and $I_l^Q \operatorname{div} v = I_l^Q \operatorname{div} \Pi_l v$, and thus

$$\|\Pi_l\|_{A_l} \leq c. \tag{55}$$

Because $Eu_1 - u_1 \in V^+$ and Π_l vanishes on V^+, it is a left-inverse to the lifting defined above. Because Π_2 preserves support in Ω_i, also Π_l maps $H_0^1(\tilde{\Omega}_i)$ into $V_{l,i}$.

Let $\{\Psi_i\}$ be a partition of unity fulfilling

$$\sum \Psi_i = 1, \qquad \operatorname{supp} \Psi_i \subset \tilde{\Omega}_i,$$

$$h^2 \|\Psi_i\|_{2,\infty} + h \|\Psi_i\|_{1,\infty} + \|\Psi_i\|_{0,\infty} \leq c.$$

The product rule and integration by parts gives

$$\|\Psi_i \varphi\|_2 \leq c\, (h^{-2} \|\varphi\|_{0,\Omega_i} + \|\varphi\|_{2,\Omega_i}). \tag{56}$$

Using $\Pi_l \operatorname{rot}(\Psi_i \varphi) \in V_{l,i}$, (55), (56), we get

$$\begin{aligned} \|\Pi_l \operatorname{rot} \varphi\|_D^2 &\leq \sum \|\Pi_l \operatorname{rot}(\Psi_i \varphi)\|_{A_l}^2 \leq c \sum \|\operatorname{rot}(\Psi_i \varphi)\|_{A_l}^2 \leq c \sum \|\Psi_i \varphi\|_2^2 \\ &\leq \ c \sum \left(h^{-4} \|\varphi\|_{0,\Omega_i}^2 + \|\varphi\|_{2,\Omega_i}^2 \right) \leq c \left(h^{-4} \|\varphi\|_0^2 + \|\varphi\|_2^2 \right) \end{aligned}$$

By local L_2-projection onto a C^1 continuous FE-space of the same mesh-size h_l we can split $\varphi = \varphi_l + \tilde{\varphi}$ such that

$$\|\varphi_l\|_j \leq c\, \|\varphi\|_j \qquad j = 0, 1, 2,$$

and the inverse inequality

$$\|\varphi_l\|_2 \leq c h^{-2} \|\varphi_l\|_0$$

and the approximation inequality

$$\|\tilde{\varphi}\|_0 \leq c h^2 \|\tilde{\varphi}\|_2$$

are fulfilled. This gives $\|\Pi_l \operatorname{rot} \varphi_l\|_{D_l} \leq c h^{-2} \|\varphi_l\|_0$ and $\|\Pi_l \operatorname{rot} \tilde{\varphi}\|_{D_l} \leq c\, \|\tilde{\varphi}\|_2$. Using operator interpolation and norm equivalence $H^1(\Omega) \sim [L_2(\Omega), H^2(\Omega)]_{1/2}$ we get

$$\begin{aligned} \|u_1\|_{[D_l, A_l]_{1/2}} &= \|\Pi_l \operatorname{rot} \varphi\|_{[D_l, A_l]_{1/2}} \leq \|\Pi_l \operatorname{rot} \varphi_l\|_{[D_l, A_l]_{1/2}} + \|\Pi_l \operatorname{rot} \tilde{\varphi}\|_{[D_l, A_l]_{1/2}} \\ &\leq \ c \left(\|\varphi_l\|_{[h^{-2} L_2, H^2]_{1/2}} + \|\tilde{\varphi}\|_{H^2} \right) \leq c\, (\|\varphi_l\|_{h^{-1} H^1} + \|\tilde{\varphi}\|_{H^2}) \\ &\leq \ c \left(h^{-1} \|\varphi\|_1 + \|\varphi\|_2 \right) \leq c h^{-1} \|u_1\|_0 \leq c\, \|u_1\|_{l,0}. \end{aligned}$$

□

The component u_2 is orthogonal to divergence free functions and has non-smooth divergence.

Lemma 8
Let u_2 be defined in (49). Then the estimate

$$\|u_2\|_{[D_l,A_l]_{1/2}} \leq c\,\|u_2\|_{l,0} \tag{57}$$

is valid.

Proof: We use $\|.\|^2_{A_l} \leq c\,\|.\|^2_{D_l}$, $\|.\|^2_{A_l} \leq ch^{-2}\varepsilon^{-1}\|.\|^2_0$ and the intermediate result (51) to obtain

$$\begin{aligned}\|u_2\|^2_{[D_l,A_l]_{1/2}} &\leq c\,\|u_2\|^2_{D_l} = c \inf_{u_2=\sum u_i} \sum \|u_i\|^2_{A_l} \\ &\leq c \inf_{u_2=\sum u_i} h^{-2}\varepsilon^{-1}\|u_i\|^2_0 \leq ch^{-2}\varepsilon^{-1}\|u_2\|^2_0 \leq c\,\|u\|^2_{l,0}.\end{aligned}$$

□

The part u_3 with smooth divergence will now be estimated by better approximation of the coarse grid interpolant of the dual variable.

Lemma 9
Let u_3 be defined in (49). Then the estimate

$$\|u_3\|_{[D_l,A_l]_{1/2}} \leq c\,\|u_3\|_{l,0} \tag{58}$$

is valid.

Proof: By definition of u_3 we have $I_l^Q \operatorname{div} u_3 = I_{l-1}^Q \operatorname{div} u$, and together with stability of X_{l-1} we get $\|u_3\|_1 \leq \|I_{l-1}^Q \operatorname{div} u\|_0$. This gives

$$\|u_3\|^2_{A_l} \leq c\,(\|u_3\|^2_1 + \varepsilon^{-1}\|I_l^Q \operatorname{div} u_3\|^2_0) \leq c\varepsilon^{-1}\|I_{l-1}^Q \operatorname{div} u\|^2_0 \leq c\,\varepsilon\,\|u_3\|^2_{l,0}.$$

On the other hand, we have

$$\|u_3\|^2_{D_l} \leq c \inf_{u_3=\sum u_i} h^{-2}\varepsilon^{-1}\|u_i\|^2_0 \leq c\,\varepsilon^{-1}\,h^{-2}\,\|u_3\|^2_0 \leq c\,\varepsilon^{-1}\,\|u_3\|^2_{l,0}.$$

By operator interpolation we finish the proof.

□

7 Numerical Results

Several versions of the multigrid method in primal variables developed and analyzed above have been tested numerically. The following two problems have been investigated within the finite element code FEPP on a SUN Ultra 1 / 166 MHz workstation with 320 MB RAM.

Problem A: Driven Cavity example.
We consider the unit square $\Omega = (0,1)^2$. The initial triangulation $\mathcal{T}_1$ is given by two triangles, further meshes are obtained by successive refinement. We have used the finite element space based on P_2 elements. The bilinear-form $A_L(.,.)$ on the finest level is defined in (8), where the projection I_L^Q maps into the piece-wise constant FE-space. The source term is set to $f = 0$. Dirichlet boundary conditions are specified as

$$u_L = \begin{cases} (1,0)^T & \text{at nodes } \in [0,1]\times\{1\}, \\ (0,0)^T & \text{at nodes } \notin [0,1]\times\{1\}, \end{cases}$$

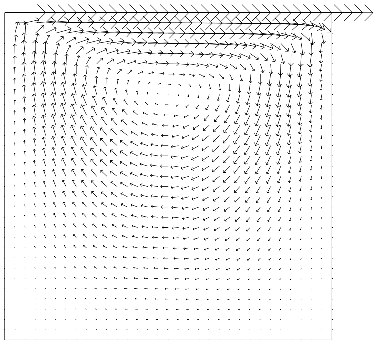


Figure 4: Solution of Problem A

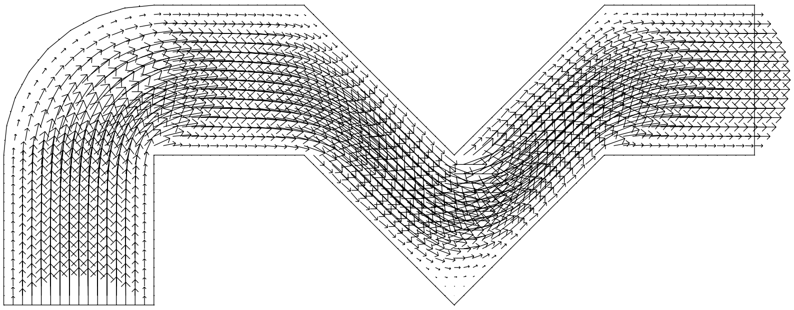


Figure 5: Solution of Problem B

and incorporated by homogenization of the FE system. A plot of the solution at level 5 is given in Figure 4.

Problem B: Flow through a pipe.
The geometry and the solution at level 4 are given in Figure 5. The boundary is split into the jacket Γ_1, inlet boundary Γ_2 and outlet boundary Γ_3. We specify homogeneous Dirichlet boundary conditions at Γ_1 and natural boundary conditions elsewhere. We solve the finite element problem find $u_L \in V_L$ such that

$$\tilde{A}_L(u_L, v_L) = (g, v_L)_{0,\Gamma_2} \qquad \forall\, v_L \in V_L.$$

The bilinear-form $\tilde{A}(u,v)$ is obtained from $A(u,v)$ by replacing the term $(e(u), e(v))_0$ by $(\nabla u, \nabla v)_0$. This is done to obtain physically correct boundary conditions. The boundary stress is defined as $g = (0,1)^T$. The problem involves curved boundary approximation, a non-convex domain and mixed boundary conditions.

At first, we investigate the behavior of the condition number $\kappa(C_L^{-1}A_L)$ in dependence of the number of levels L and the parameter ε. The preconditioner C_L is obtained by the application of a symmetric multigrid operator, either a W-2-2 cycle or a V-1-1 cycle. In addition to the additive smoother (16), we use the multiplicative counterpart

$$S_l^A = \prod_{i=1}^{n_l} (I - P_{l,i}^{A_l}), \tag{59}$$

for pre-smoothing and in reversed order for post-smoothing. It does not need damping at all. The numerical results for the condition number $\kappa(C_L^{-1}A_L)$ for Problem A obtained by

l	Unknowns	additive smoother				multiplicative smoother			
	$\varepsilon =$	10^0	10^{-2}	10^{-4}	10^{-6}	10^0	10^{-2}	10^{-4}	10^{-6}
2	50	1.82	2.51	2.66	2.66	1.04	1.10	1.11	1.11
3	162	2.27	6.79	7.66	7.67	1.26	2.15	2.29	2.30
4	578	2.58	8.59	9.91	9.93	1.37	2.47	2.64	2.64
5	2178	2.72	9.79	11.60	11.62	1.39	2.56	2.73	2.73
6	8450	2.79	10.84	13.12	13.15	1.39	2.65	2.82	2.82
7	33282	2.73	11.66	14.41	14.45	1.39	2.72	2.90	2.91

Table 1: Condition numbers for V-1-1 cycle

l	Unknowns	additive smoother				multiplicative smoother			
	$\varepsilon =$	10^0	10^{-2}	10^{-4}	10^{-6}	10^0	10^{-2}	10^{-4}	10^{-6}
2	50	1.05	1.08	1.10	1.10	1.000	1.00	1.00	1.00
3	162	1.15	1.65	1.74	1.74	1.002	1.05	1.06	1.06
4	578	1.19	1.76	1.73	1.73	1.002	1.05	1.05	1.06
5	2178	1.24	1.79	1.87	1.86	1.002	1.04	1.05	1.05
6	8450	1.26	1.86	1.92	1.91	1.002	1.05	1.05	1.05
7	33282	1.26	1.87	1.92	1.92	1.002	1.05	1.05	1.05

Table 2: Condition numbers for W-2-2 cycle

the Lanzcos method are given in Table 1 for a V-1-1 cycle and in Table 2 for a W-2-2 cycle. For the W-2-2 the calculated condition numbers neither depend on the level nor on the parameter, what is in correspondence with the analysis provided. We do not have optimal estimates for V-cycle convergence rate yet, but the numerical results seem to be very promising.

Next, we used the V-1-1 multigrid preconditioner in a preconditioned conjugate gradients solver for the solution of Problem A and Problem B. The small parameter is set to $\varepsilon = 10^{-6}$. The iteration is terminated after an reduction of the error in energy norm by a factor of 10^8. The necessary iteration numbers and CPU times are shown in Table 3 and Table 4, respectively.

Level	Unknowns	Iterations	Time[sec]
2	50	4	0.01
3	162	10	0.08
4	578	15	0.41
5	2178	15	1.88
6	8450	16	8.56
7	33282	16	37.06
8	132098	16	154.80

Table 3: Iteration numbers and CPU times for Problem A, PCG with V-1-1 cycle

Level	Unknowns	Iterations	Time[sec]
2	230	10	0.1
3	810	13	0.6
4	3026	15	2.7
5	11682	17	12.9
6	45890	18	58.2
7	181890	18	242.0

Table 4: Iteration numbers and CPU times for Problem B, PCG with V-1-1 cycle

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