Robust Multigrid Preconditioning for Parameter-Dependent Problems I: The Stokes-type Case

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Abstract

Parameter dependent problems can be discretized by selective reduced integration methods to achieve parameter independent discretization errors. The convergence rate of standard multigrid solvers applied to the primal linear system deteriorates, if the parameter becomes small. In this paper, we construct multigrid components leading to parameter independent rates. We need a robust base iteration as smoother and uniformly continuous grid transfer operations. The suggested multigrid preconditioner is applied to problems from linear elasticity.

1 Introduction

Several problems in computational mechanics lead to equations of the form

$$\left(A + \frac{1}{\varepsilon} B\right) u = f, \tag{1}$$

with a well conditioned operator A, an operator B with non-trivial null-space, and a small, positive parameter ε. If we are interested in the behavior of the solution for $\varepsilon \to 0$, we call (1) a 'parameter dependent problem' [1]. Usually, the three steps stability analysis, discretization and equation solving are robustly handled within the framework of mixed finite element methods [8]. There are two different approaches to solve the linear system, either by block transformations using standard multigrid preconditioners [11], [6], or by multigrid methods directly applied to the indefinite system [4], [7].

A great simplification for practical implementation was achieved by selective reduced integration (SRI) methods. In many cases, a SRI - discretization scheme is equivalent to a mixed finite element discretization [12], [1]. By the equivalence, stabilty and robust discretization hold for the primal problem, but a robust iterative solver was not available. Several authors reported parameter dependent iteration numbers for multigrid solvers with

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classical components. The aim of this paper is to introduce robust multigrid components needed for parameter independent multigrid rates. More precisely, we suggest a robust base iteration to be used as smoother and uniformly continuous grid transfer operations.

In §2 we specify the abstract framework and give some examples, in §3 we give the finite element discretization. The next three sections are devoted to the development of the multigrid method, in §4 we construct the smoother, in §5 we introduce the grid transfer operations and in §6 norms for the smoothing and approximation property are suggested. Numerical examples are given in §7.

2 Problem Formulation

The parameter dependent problem is considered in an abstract framework. Therefore let V be some Hilbert-space with dual V^*, norm $\|.\|$, inner product $(.,.)$ and duality product $\langle .,. \rangle$. The bilinearforms $a(.,.)$ and $b(.,.)$ and the associated linear operators $\langle Au, v\rangle := a(u,v)$ and $\langle Bu, v\rangle := b(u,v)$ are symmetric and continuous

$$\begin{array}{rclcll} a(u,v) & = & a(v,u) & \leq & \overline{\alpha}\|u\|\,\|v\| & \forall u,v \in V, \\ b(u,v) & = & b(v,u) & \leq & \overline{\beta}\|u\|\,\|v\| & \forall u,v \in V. \end{array} \tag{2}$$

We assume that $a(u,u)$ and $b(u,u)$ are non-negative and the operator B has non-trivial null-space $V_0 \subset V$. Further, we assume ellipticity conditions on subspaces

$$\begin{array}{rcll} a(u,u) & \geq & \underline{\alpha}\|u\|^2 & \forall u \in V_0, \\ b(u,u) & \geq & \underline{\beta}\|u\|^2 & \forall u \in V_0^{\perp}. \end{array} \tag{3}$$

The ellipticity condition on b is fulfilled for regular problems of Stokes-type, but is violated for singular problems of Mindlin-Reissner-type.

With the right-hand side $f \in V^*$ we are looking for a solution $u \in V$ fulfilling (1) in V^*. When we refer to the standard-conditions, we mean (2) and (3). The proofs of the following theorems are given in [13].

Theorem 1 *Assume the standard conditions are fulfilled. Then, for the solution u of Problem (1) we have*

$$\|u\|_V \leq c\|f\|_{V^*}$$

with a constant c depending on $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$, $\overline{\beta}$, but not depending on $\varepsilon \in (0,1]$.

Theorem 2 *Let $u(\varepsilon), \varepsilon > 0$ be the solution of (1) and let $u(0)$ be the solution of the null-space problem*

$$\text{find } u \in V_0 : \quad a(u,v) = \langle f, v\rangle \quad \forall v \in V_0. \tag{4}$$

Then there holds for $\varepsilon \to 0$

$$u(\varepsilon) \to u(0).$$

The standard conditions have to be checked for the individual examples. Often, they follow from stability conditions of equivalent mixed problems. Assume we have some, non unique multiplicative splitting of B in the form

$$B = \overline{B}^* C^{-1} \overline{B}, \tag{5}$$

with the operator $\overline{B}$ from V into the dual M^* of some Hilbert-space M, its dual operator $\overline{B}^*$, and the symmetric, positive, invertible operator C from M into M^*. Then the solution u together with $p := \varepsilon^{-1} C^{-1} \overline{B} u$ is a solution of the mixed problem

$$\begin{pmatrix} A & \overline{B}^* \\ \overline{B} & -\varepsilon C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{6}$$

We can either conclude stability by the equivalent mixed problem, or we can derive the standard conditions from conditions proved for the mixed problem and use Theorem 1.

Theorem 3 *Assume $\overline{B}$ fulfills the LBB - condition*

$$\forall q \in M \;\; \exists u \in V : \quad \left\langle \overline{B} u, q \right\rangle_{M^* \times M} \geq \beta_1 \|u\|_V \|q\|_M$$

and C is continuous

$$\langle Cp, q \rangle_{M^* \times M} \leq \gamma \|p\|_M \|q\|_M \quad \forall p, q \in M.$$

Then B is elliptic on $V_0^\perp$ with constant

$$\underline{\beta} = \frac{\beta_1^2}{\gamma}$$

Example 1 (Nearly incompressible materials) *Let Ω be a convex polygonal domain in $\mathbf{R}^2$ with $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$ s.t. $meas\{\Gamma_D\} \neq 0$. $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}^2$. Consider Lame's equation in weak form*

$$2\mu \underbrace{\int\limits_\Omega \sum_{i,j} \varepsilon_{ij}(u) \varepsilon_{ij}(v) \, dx}_{\langle Au, v \rangle} + \lambda \underbrace{\int\limits_\Omega div\, u \; div\, v \, dx}_{\langle Bu, v \rangle} =$$

$$= \underbrace{\int\limits_\Omega f^T v \, dx + \int\limits_{\Gamma_N} g^T v \, ds}_{2\mu \langle f, v \rangle} \qquad \forall v \in V$$

with the symmetric derivatives $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, the source term $f \in [L_2(\Omega)]^2$ and the boundary traction $g \in [L_2(\Gamma_N)]^2$. If the Poisson ration ν tends to 0.5, i.e. the material becomes incompressible, the ratio λ/μ tends to ∞. Therefore we define the small parameter $\varepsilon := 2\mu/\lambda$. The null-space V_0 of B consists of the divergence free functions in V. The standard conditions are fulfilled [8], Theorems 3 and 1.

Example 2 (Timoshenko beam problem) *Let I be a finite interval in $\mathbf{R}$, $V = [H_0^1(I)]^2$. Find $(w,\Theta) \in V$ such that*

$$\underbrace{\int_I \Theta' \Psi' \, dx}_{\langle A(w,\Theta),(v,\Psi)\rangle} + \frac{1}{t^2} \underbrace{\int_I (w' - \Theta)(v' - \Psi)\, dx}_{\langle B(w,\Theta),(v,\Psi)\rangle} = \underbrace{\int_I f v \, dx}_{\langle f,(v,\Psi)\rangle} \quad \forall (v,\Psi) \in V$$

Here w is the vertical displacement, Θ is the rotation of the vertical fiber and f is the vertical force. The small parameter t is the thickness of the beam. The null-space V_0 consists of pairs (w,Θ) fulfilling the weak form of the Bernoulli hypothesis $\Theta = w'$. The standard conditions are fulfilled [1].

Example 3 (Mindlin Reissner plate problem) *The two dimensional analogon to the Timoshenko Beam is the Mindlin Reissner plate. Let Ω be a convex polygonal domain in $\mathbf{R}^2$, $V = [H_0^1(\Omega)]^2 \times H_0^1(\Omega)$. Find $(w,\Theta) \in V$ such that*

$$\underbrace{\int_\Omega \left[(1-\nu)\varepsilon(\Theta) : \varepsilon(\Psi) + \nu \, div \, u \; div \, v\right] \, dx}_{\langle A(w,\Theta),(v,\Psi)\rangle} +$$

$$+\frac{1}{t^2} \underbrace{\int_\Omega (\nabla w - \Theta)(\nabla v - \Psi)\, dx}_{\langle B(w,\Theta),(v,\Psi)\rangle} = \underbrace{\int_\Omega f v \, dx}_{\langle f,(v,\Psi)\rangle}.$$

Although the formal structure looks the same, non-trivial difficulties are hidden in the operator B, which does not fulfill the required ellipticity condition (3). This is connected to the non-continuous bilinearform c of the saddle point problem [3]. We exclude the Mindlin Reissner plate from our following investigations.

3 Discretization Techniques

A straight forward finite element discretization of (1) leads to a parameter-dependent discretization error. Cea's lemma provide an upper bound of order ε^{-1}, which is no shortcoming of the analysis, but can be also found in practical computations. This 'phenomenon' is well known as locking [12], [1], [2].

Possible ways out are methods of selective reduced integration (SRI), which can be seen as non-conforming discretization methods. We will use standard finite element subspaces $V_h \subset V$, but we replace the bilinearform $b(.,.)$ by some mesh-dependent bilinearform $b_h(.,.)$. The discrete problem is to find $u_h \in V_h$ such that

$$a(u_h, v_h) + \frac{1}{\varepsilon} b_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h. \tag{7}$$

Using the finite element basis $\{p_i\}$ we get the linear system

$$\left(A + \frac{1}{\varepsilon} B\right) u = f,$$

or simply

$$\mathcal{A} u = f$$

with the definition $\mathcal{A} = A + \varepsilon^{-1} B$. In general, the null-space $V_{0h} \subset V_h$ of b_h is not contained in V_0, and therefore we need the additional discrete standard conditions

$$\begin{array}{rcll} a(u_h, u_h) & \geq & \underline{\alpha}_h \|u_h\|^2 & \forall u_h \in V_{0h}, \\ b(u_h, u_h) & \geq & \underline{\beta}_h \|u_h\|^2 & \forall u_h \in V_{0h}^{\perp}. \end{array} \tag{8}$$

We also assume continuity of b_h with constant $\overline{\beta}_h$ on V_h. The discretization is locking-free, if these constants do not depend on the mesh parameter h. This is provided by the convergence theorem

Theorem 4 (FEM convergence) *Assume the standard conditions and the discrete standard conditions are fulfilled with constants independent of ε and h. Then the discretization error is bounded by*

$$\begin{aligned} \|u - u_h\|_V \leq c \quad & \Big(\inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{v_h \in V_h} \|w - v_h\|_V + \\ & \|(B - B_h) u\|_{V_h^*} \Big) \end{aligned}$$

with the solution $w \in V_0^{\perp}$ of the auxiliary problem

$$b(w, v) = \langle f, v \rangle - a(u, v) \quad \forall v \in V_0^{\perp}.$$

The constant c depends only on $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\alpha}_h$, $\underline{\beta}$, $\overline{\beta}$, $\underline{\beta}_h$, $\overline{\beta}_h$, but not on the parameter ε and on the mesh size h.

An alternative technique to prove discrete convergence is to use convergence results for equivalent mixed finite element schemes. Now we will apply this discretization technique to the examples from §2.

Example 1 is discretized by the finite element space

$$V_h = \left\{ v_h \in V : v_h|_T \in [P_2(T)]^2 \;\; \forall T \in \mathcal{T}_h \right\} \tag{9}$$

on the regular triangulation $\mathcal{T}_h = \{T\}$ of Ω (in the sense of [9]) and the mesh dependent bilinearform

$$b_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T \overline{\operatorname{div} u_h} \; \overline{\operatorname{div} v_h} \, \mathrm{dx} \tag{10}$$

with the L_2 - projection $\overline{\operatorname{div} u_h}$ of $\operatorname{div} u_h$ onto the element-wise constant functions. This discretization is equivalent to the mixed discretization by piece-wise quadratic functions for u_h and piece-wise constants for the pressure p_h. This pairing fulfills the discrete LBB - condition [8] from which we derive the discrete standard conditions. The assembling of the element-stiffness matrices is implemented by mid-point selective reduced integration. The functions in V_{0h} are characterized by

$$\int\limits_T \operatorname{div} u_h \,\mathrm{dx} = \int\limits_{\partial T} n^T u_h \,\mathrm{ds} = 0, \qquad \forall T \in \mathcal{T}. \tag{11}$$

Example 2 is discretized by the finite element space

$$V_h = \left\{ v_h \in V : v_h|_T \in [P_1(T)]^2 \;\; \forall T \in \mathcal{T}_h \right\}$$

on the discretization $\mathcal{T}_h = \{T\}$ of the interval I and the mesh dependent bilinearform

$$b_h((w_h, \Theta_h), (v_h, \Psi_h)) := \sum_{T \in \mathcal{T}} \int\limits_T (\overline{w_h' - \Theta_h})(\overline{v_h' - \Psi_h}) \mathrm{dx}.$$

This discretization is equivalent to the mixed finite element discretization using piece-wise linear elements for the primal functions (w_h, Θ_h) and piece-wise constants for the dual shear term, fulfilling the discrete LBB condition [1]. An element $(w_h, \Theta_h) \in V_{0h}$ fulfills the discrete Bernoulli condition if

$$\int\limits_T (w_h' - \Theta_h) \,\mathrm{dx} = 0, \qquad \forall T \in \mathcal{T}. \tag{12}$$

4 Robust Smoothers

We construct a robust iterative method for parameter dependent problems to be used as smoothing iteration in a multigrid method. For motivation, we dissect a thin domain Ω into one layer of thin elements as seen in Figure 1. The small parameter is the thickness t, which can be much smaller than the element length h. By finite differences discretization of the Laplace operator we get the matrix

$$\mathcal{A} = \begin{pmatrix}
\frac{1}{h^2}+\frac{1}{t^2} & -\frac{1}{t^2} & -\frac{1}{h^2} & & & & & \\
-\frac{1}{t^2} & \frac{1}{h^2}+\frac{1}{t^2} & & -\frac{1}{h^2} & & & & \\
-\frac{1}{h^2} & & \frac{2}{h^2}+\frac{1}{t^2} & & \ddots & & & \\
& -\frac{1}{h^2} & -\frac{1}{t^2} & \frac{2}{h^2}+\frac{1}{t^2} & & \ddots & & \\
& & \ddots & & \ddots & \ddots & -\frac{1}{h^2} & \\
& & & \ddots & \ddots & \ddots & & -\frac{1}{h^2} \\
& & & & -\frac{1}{h^2} & & \frac{1}{h^2}+\frac{1}{t^2} & -\frac{1}{t^2} \\
& & & & & -\frac{1}{h^2} & -\frac{1}{t^2} & \frac{1}{h^2}+\frac{1}{t^2}
\end{pmatrix}.$$

We set $\varepsilon = t^2$ and split $\mathcal{A} = A + \varepsilon^{-1} B$. The matrix A consists of two copies of the one dimensional Laplace discretization, while B is block-diagonal with the 2×2 blocks

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

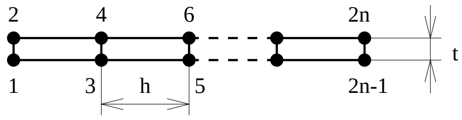

Figure 1: Thin domain

The condition number of $\mathcal{A}$ is $O(h^{-2}+\varepsilon^{-1})$, which implies an ε and h dependent convergence rate of the Richardson iteration. When we apply 2×2 block-diagonal preconditioning, the generalized condition number is $O(h^{-2})$. It is robust in terms of the small parameter ε, but of course it depends on the mesh-size h. Block preconditioning for anisotropic meshes is investigated in [5].

To prepare the abstract theorem, we formulate this example in an general setting. The null-space V_0 of B plays the key-role. It is spanned by the null-space basis-functions

$$q_i = (0,\ldots,0, \underset{\substack{\uparrow \\ 2i-1}}{1}, \underset{\substack{\uparrow \\ 2i}}{1}, 0,\ldots,0)^T.$$

We consider the injections

$$P_i = \begin{pmatrix} 0 & 0 \\ \cdots & \cdots \\ 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{pmatrix}$$

from $\mathbf{R}^2$ into $\mathbf{R}^{2n}$ and corresponding subspaces $V_i = R(P_i)$. Then the 2×2 block-diagonal matrix C is the preconditioner of the additive Schwarz method (ASM)

$$C^{-1} = \sum_i P_i (P_i^T \mathcal{A} P_i)^{-1} P_i^T. \tag{13}$$

The sub-spaces V_i have been chosen such that

$$\forall i \; \exists j : q_i \in V_j. \tag{14}$$

Theorem 5 (Robust smoother) *Assume, the following assumptions are fulfilled*

1. *A, B and B_h fulfill the standard conditions (2), (3) and (8).*

2. *There is given a space splitting $V = \sum V_i$, $V_i = R(P_i)$ with maximal overlap m.*

3. *For all $u \in V$ and $u_0 \in V_0$ there exist splittings*

$$u = \sum_i u_i, \qquad u_i \in V_i$$
$$u_0 = \sum_i u_{0,i}, \qquad u_{0,i} \in V_i \cap V_0$$

such that the inequalities

$$\begin{aligned} \|u\|^2 &\geq c_1(h) \sum_i \|u_i\|^2, \\ \|u_0\|^2 &\geq c_1(h) \sum_i \|u_{0,i}\|^2 \end{aligned}$$

hold.

Then the ASM preconditioner

$$C^{-1} := \sum_i P_i (P_i^T \mathcal{A} P_i)^{-1} P_i^T \tag{15}$$

fulfills the spectral inequalities

$$c\, c_1(h)\; C \leq \mathcal{A} \leq m\; C$$

with the ε and h independent constant c.

Proof: It is enough to show the spectral inequalities

$$c\, c_1(h)\; |||u|||^2 \leq \|u\|_{\mathcal{A}}^2 \leq m\; |||u|||^2$$

with the split norm

$$|||u|||^2 := \inf_{\substack{u_i \in V_i \\ u = \sum u_i}} \sum \|u_i\|_{\mathcal{A}}^2,$$

see, e.g. [15]. The right inequality is straight forward

$$\|u\|_{\mathcal{A}}^2 = \inf_{\substack{u_i \in V_i \\ u = \sum u_i}} \|\sum u_i\|_{\mathcal{A}}^2 \leq \inf_{\substack{u_i \in V_i \\ u = \sum u_i}} m \sum_i \|u_i\|_{\mathcal{A}}^2 = m\; |||u|||^2$$

To prove the left inequality, we split u according to Assumption 3

$$u = \underbrace{u_0}_{\in V_0} + \underbrace{u_1}_{\in V_0^\perp} = \sum_i u_{0,i} + \sum_i u_{1,i}.$$

$$\begin{aligned} \|u\|_{\mathcal{A}}^2 &= \|u_0 + u_1\|_A^2 + \frac{1}{\varepsilon}\|u_1\|_B^2 \\ &\geq \|u_0 + u_1\|_A^2 + \frac{\underline{\beta}_h}{2\overline{\alpha}}\|u_1\|_A^2 + \frac{\underline{\beta}_h}{2\varepsilon}\|u_1\|^2 \\ &\geq \min\left\{1, \frac{\underline{\beta}_h}{2\overline{\alpha}}\right\} \frac{1}{2}\|u_0\|_A^2 + \frac{\underline{\beta}_h}{2\varepsilon}\|u_1\|^2 \\ &\geq \frac{1}{2}\min\left\{\underline{\alpha}_h, \underline{\beta}_h, \frac{\underline{\beta}_h \underline{\alpha}_h}{2\overline{\alpha}}\right\} \left(\|u_0\|^2 + \frac{1}{\varepsilon}\|u_1\|^2\right). \end{aligned}$$

Starting with $\|u\|$ we get

$$\begin{aligned}
\|u\|^2 &\leq \sum_i \left(\|u_{0,i} + u_{1,i}\|_A^2 + \frac{1}{\varepsilon}\|u_{0,i} + u_{1,i}\|_B^2 \right) \\
&\leq 2\sum_i \left(\|u_{0,i}\|_A^2 + \|u_{1,i}\|_A^2 + \frac{1}{\varepsilon}\|u_{1,i}\|_B^2 \right) \\
&\leq 2\max\left\{\overline{\alpha}, \overline{\beta}_h\right\} \sum_i \left(\|u_{0,i}\|^2 + \|u_{1,i}\|^2 + \frac{1}{\varepsilon}\|u_{1,i}\|^2 \right) \\
&\leq 4\, c_1(h) \max\left\{\overline{\alpha}, \overline{\beta}_h\right\} \left(\|u_0\|^2 + \frac{1}{\varepsilon}\|u_1\|^2 \right),
\end{aligned}$$

which finishes the proof.

Remark 1 *The construction of sub-spaces V_i can be done due to a basis for the null-space V_0. Assume, the basis $\{q_i\}$ for V_0 fulfills the inverse inequality*

$$\| \sum_i u_i q_i \|^2 \geq c_1(h) \sum_i \|u_i q_i\|^2. \tag{16}$$

For the considered H^1 - problems, we expect an h dependency of

$$c_1(h) = ch^2.$$

Instead of working with one dimensional sub-spaces $V_i = span\{q_i\}$, the choice of larger spaces V_j, such that (14) holds, is allowed and may be much simpler to implement.

For Example 1, the three types of null-space basis functions are shown in Figure 2, see [8]. On each element T of the mesh the functions fulfill (11). These basis functions can be covered by a space splitting drawn in Figure 3. Each sub-space is associated to a vertex and contains the unknowns in the vertex and in the mid-points of the adjacent edges. On uniform meshes, the dimensions of the sub-spaces for inner nodes are 14. The inverse inequality (16) with $c_1(h) = ch^2$ is proven by standard techniques. If the domain is not simply connected, or the Neumann part of the boundary is not simply connected, we need global basis functions describing global curls. These global curls are resolved by the coarse-grid correction described later, so we don't take care of them in the smoother.

Two types of null-space basis functions for *Example 2* are drawn in Figure 4. The whole basis is generated by repeatedly shifting these two basis functions by two elements. These functions fulfill the discrete Bernoulli condition (12). The sub-spaces V_i contain the unknowns belonging to the three nodes in the support of the basis functions, see Figure 5. Again, the inverse inequality (16) holds with $c_1(h) = ch^2$.

5 Two-Grid Method

The next step towards a multigrid method is the two-grid preconditioner. Therefore we have to define a coarse grid problem and grid transfer operators. First, we use these

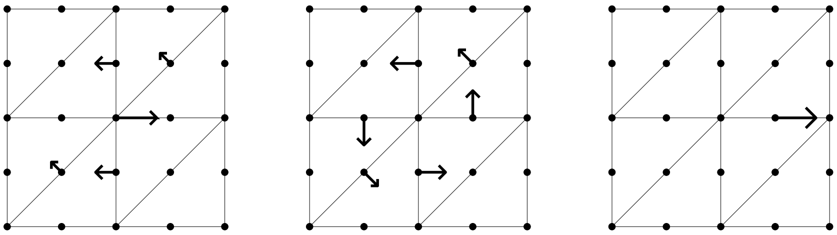

Figure 2: div-free null-space basis functions

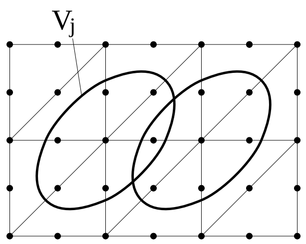

Figure 3: Spaces containing div-free basis functions

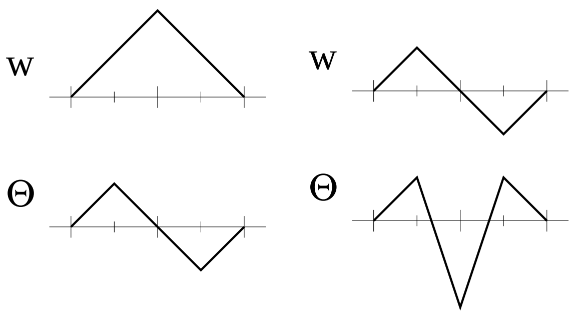

Figure 4: Bernoulli null-space basis functions

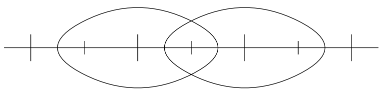

Figure 5: Spaces containing Bernoulli basis functions

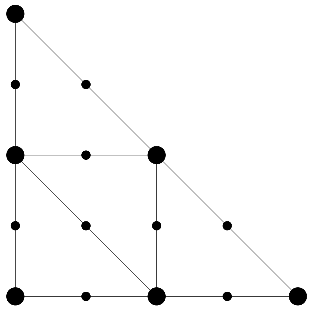

Figure 6: Refinement of coarse-grid triangle T

components in an additive two-grid method with the preconditioner

$$C^{-1} := I_H^h \mathcal{A}_H^{-1} (I_H^h)^T + C_{SM}^{-1}. \tag{17}$$

The most natural way to define the coarse grid matrix $\mathcal{A}_H$ is to use a non-conforming method on the coarse grid [4]. We denote the prolongation by I_H^h and apply the transposed matrix as restriction. C_{SM} is the ASM - smoother defined in (15).

For the analysis we use the split-norm with preconditioning

$$|||u|||^2 := \inf_{\substack{u = I_H^h u_H + \sum u_i \\ u_H \in V_H,\, u_i \in V_i}} \|u_H\|_{\mathcal{A}_H}^2 + \sum \|u_i\|_{\mathcal{A}_h}^2. \tag{18}$$

Again, the norm equivalence of $|||.|||$ and $\|.\|_{\mathcal{A}_h}$ proves the preconditioner spectral equivalence inequalities with the same constants.

The prove of (18) depends on the underlying problem, but in general we need a spectrally equivalent coarse grid problem

$$\underline{c}\, \|u_H\|_{\mathcal{A}_H}^2 \leq \|I_H^h u_H\|_{\mathcal{A}_h}^2 \leq \overline{c}\, \|u_H\|_{\mathcal{A}_H}^2 \tag{19}$$

with ε and h independent constants $\underline{c}$ and $\overline{c}$. By the choice of $u_H \in N(B_H)$ we get

$$\|I_H^h u_H\|_{\mathcal{A}_h}^2 = \|I_H^h u_H\|_A^2 + \frac{1}{\varepsilon} \|I_H^h u_H\|_{B_h}^2 \leq \overline{c} \|u_H\|_{\mathcal{A}_H}^2 = \overline{c} \|u_H\|_{A_H}^2,$$

which can be true only if

$$I_H^h u_H \in N(B_h)$$

holds, at least up to $O(\varepsilon)$. For Example 1 this means, that coarse grid divergence free functions must be prolongated to fine grid divergence free functions. For the SRI discretization from above this prolongation can be constructed as follows. A coarse grid triangle T is divided into four fine-grid triangles $t_i, i = 1, \ldots 4$ as seen in Figure 6. A coarse grid divergence free function $u_H \in N(B_H)$ fulfills

$$\int\limits_{\partial T} n^T u_H \mathrm{ds} = 0, \quad \forall T \in \mathcal{T}_H, \tag{20}$$

the prolongation $I_H^h u_H$ has to be fine-grid divergence free in the form

$$\int\limits_{\partial t} n^T I_H^h u_H \mathrm{ds} = 0, \qquad \forall t \in \mathcal{T}_h. \tag{21}$$

Be using natural, i.e. quadratic prolongation on ∂T, we get the relation

$$\sum_{t_i \subset T} \int\limits_{\partial t} n^T I_H^h u_H \mathrm{ds} = 0, \quad \forall T \in \mathcal{T}_H. \tag{22}$$

The values of the three inner nodes of a coarse-grid triangle T are chosen such that (21) holds for the three triangles in the corners of T. By combining (20) and (22) we get (21) also for the inner fine grid triangles. The implementation is done by solving 6×6 sub-problems belonging to the 3 inner nodes of each coarse grid triangle.

By standard transformation arguments one gets continuity of I_H^h with respect to the energy-norms $\|.\|_{\mathcal{A}_H}$ and $\|.\|_{\mathcal{A}_h}$.

To sketch the prove of (18) for Example 1, we give an explicit construction of the splitting. First we decompose u_h into coarse-grid and fine-grid components using the continuous coarse-grid interpolation Π_H

$$u_h = I_H^h \Pi_H u_h + (I - I_H^h \Pi_H) u_h.$$

We define Π_H similar to the proof of the discrete LBB - condition for the $P_2 - P_0$ pairing [8] by

$$\Pi_H u_h(x_i) = u_h(x_i)$$
$$\int\limits_{E_i} \Pi_H u_h \mathrm{ds} = \int\limits_{E_i} u_h \mathrm{ds}$$

with the coarse grid corner nodes x_i and the coarse grid edges E_i. The fine grid complement $(I - I_H^h \Pi_H) V_h$ is split into the spaces V_i belonging to the fine grid corners y_i, which are no coarse grid corners. By the properties of the fine grid space and standard finite element transformation techniques we conclude the proof.

6 Multigrid Method

The h-independent convergence rate of the ASM preconditioner implies also h-independent convergence rate of the 2-level multiplicative Schwarz preconditioner, which is the classical two-grid method. But, by this technique we do not get the dependency of the convergence rate on the number of smoothing steps, which is necessary for the multigrid method.

To use the standard W-cycle proof we have to check the smoothing property and the approximation property with respect to proper norms [10]. We conjecture, the ASM smoother $S := I - C_{SM}^{-1} \mathcal{A}$ fulfills the smoothing property

$$\|S^\nu u_h\|_{\mathcal{A}_h} \leq \frac{c}{h\sqrt{\nu}} \|u_h\|_{\tilde{0}}, \tag{23}$$

Level	Nodes	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
2	25	1.82	2.51	2.66	2.66
3	81	2.27	6.79	7.66	7.67
4	289	2.58	8.59	9.91	9.93
5	1089	2.72	9.79	11.60	11.62
6	4225	2.79	10.84	13.12	13.15
7	16641	2.73	11.66	14.41	14.45

Table 1: Condition numbers for additive V-1-1 cycle,
nearly incompressible materials

with the scaled, L_2-like norm

$$\|u_h\|_{\tilde{0}}^2 := \|u_h\|_0^2 + \frac{h^2}{\varepsilon}\|u_h\|_{B_h}^2$$

motivated by the norm $\|u_h\|_0^2 + h^2\|p\|_0^2$ used by [14], [7] for the mixed multigrid scheme. Up to now, we could not proof the smoothing property (23), but numerical examples show that suspected behavior. The approximation property

$$\|u_h - I_H^h \mathcal{A}_H^{-1} I_h^H \mathcal{A}_h u_h\|_{\tilde{0}} \leq ch\|u_h\|_{\mathcal{A}_h}$$

follows from the approximation property of the equivalent mixed coarse-grid correction step. Note, that the pure algebraic technique would deliver the smoothing property $\|S^\nu u_h\|_{\mathcal{A}_h} \leq c\sqrt{\nu}^{-1}\|u_h\|_{C_{SM}}$, but the approximation property does not hold for this choice.

7 Examples

Finally, we give numerical results for the case of nearly incompressible materials and the Timoshenko beam problem. First, we discretize Example 1 by (9) and (10). We examine the preconditioner for the Dirichlet problem on the unit square $\Omega = [0,1]^2$. The relative condition number $\kappa(C^{-1}\mathcal{A})$ for the described multi-grid preconditioner is measured. The preconditioner is applied using a $V-1-1$ cycle and a $W-2-2$ cycle. The results for different values of the small parameter ε are shown in Table 1 and Table 2. Both regimes show robustness with respect to the small parameter. From these results it is not clear, whether the condition number of the V-cycle slightly depends on the level, but the W cycle shows a bounded condition number in terms of h.

Instead of applying the smoother in an additive framework, we can run it also multiplicatively. The condition numbers for symmetric $V-1-1$ and symmetric $W-2-2$ cycles with multiplicative smoothers are shown in Table 3 and Table 4, respectively.

The relative condition numbers of the multiplicative preconditioner applied to the Timoshenko beam model on the interval $[0,1]$ are given in Table 5. For this one dimensional

Level	Nodes	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
2	25	1.05	1.08	1.10	1.10
3	81	1.15	1.65	1.74	1.74
4	289	1.19	1.76	1.73	1.73
5	1089	1.24	1.79	1.87	1.86
6	4225	1.26	1.86	1.92	1.91
7	16641	1.26	1.87	1.92	1.92

Table 2: Condition numbers for additive W-2-2 cycle,
nearly incompressible materials

Level	Nodes	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
2	25	1.04	1.10	1.11	1.11
3	81	1.26	2.15	2.29	2.30
4	289	1.37	2.47	2.64	2.64
5	1089	1.39	2.56	2.73	2.73
6	4225	1.39	2.65	2.82	2.82
7	16641	1.39	2.72	2.90	2.91

Table 3: Condition numbers for multiplicative V-1-1 cycle,
nearly incompressible materials

Level	Nodes	$\varepsilon = 1$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
2	25	1.000	1.00	1.00	1.00
3	81	1.002	1.05	1.06	1.06
4	289	1.002	1.05	1.05	1.06
5	1089	1.002	1.04	1.05	1.05
6	4225	1.002	1.05	1.05	1.05
7	16641	1.002	1.05	1.05	1.05

Table 4: Condition numbers for multiplicative W-2-2 cycle,
nearly incompressible materials

Level	Nodes	$t=1$	$t=0.1$	$t=0.01$	$t=0.001$
3	5	1.000	1.00	1.00	1.00
4	9	1.002	1.11	1.22	1.22
5	17	1.002	1.13	1.30	1.30
6	33	1.002	1.14	1.34	1.35
7	65	1.002	1.14	1.36	1.38
8	129	1.002	1.14	1.36	1.40
9	257	1.002	1.14	1.36	1.42
10	513	1.002	1.14	1.36	1.42
11	1025	1.002	1.14	1.36	1.43
12	2049	1.002	1.14	1.36	1.43
13	4097	1.002	1.14	1.36	1.43
14	8193	1.002	1.14	1.36	1.43
15	16385	1.002	1.14	1.36	1.43

Table 5: Condition numbers for multiplicative V-1-1 cycle, Timoshenko beam model

problem we can compute more levels, such that we can see a bounded condition number in the thickness t and mesh-size h also for the $V-1-1$ cycle.

Because the solution of the nearly incompressible problem converges to the solution of the limit problem, namely Stokes problem, we can approximate the later one by a penalty method. We give two examples, one is the driven cavity problem at the unit-square, see Figure 7, the other is the flow through a pipe, Figure 8. The second example has curved boundaries and concave corners.

For both examples we set the penalty parameter ε to 10^{-6}. We used a $V-1-1$ multigrid preconditioner with multiplicative smoother in a cg-iteration. The iteration was terminated after an error reduction in preconditioner energy norm by a factor of 10^{-8}. The necessary iteration numbers and CPU times are shown in Table 6 and Table 7. The tests were done using the C++ code FEPP on a SUN Ultra 1 workstation. The solutions are plotted in Figure 9 and Figure 10.

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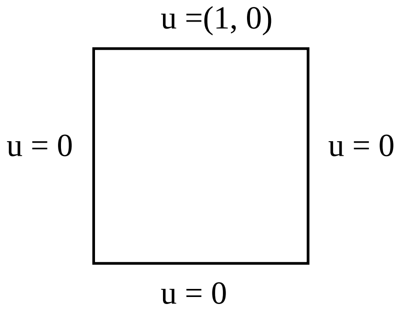

Figure 7: Driven Cavity problem

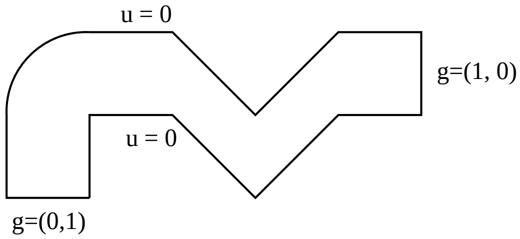

Figure 8: Pipe-flow problem

Level	Nodes	Iterations	Time[sec]
2	25	4	0.01
3	81	10	0.08
4	289	15	0.41
5	1089	15	1.88
6	4225	16	8.56
7	16641	16	37.06
8	66049	16	154.80

Table 6: Iteration numbers and CPU times for Driven Cavity

Level	Nodes	Iterations	Time[sec]
2	115	10	0.1
3	405	13	0.6
4	1513	15	2.7
5	5841	17	12.9
6	22945	18	58.2
7	90945	18	242.0

Table 7: Iteration numbers and CPU times for pipe-flow

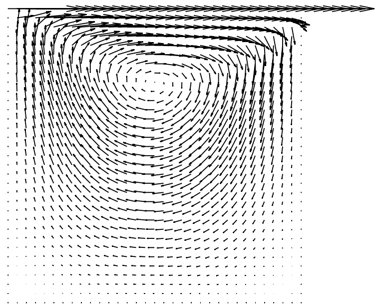

Figure 9: Solution of Driven Cavity problem

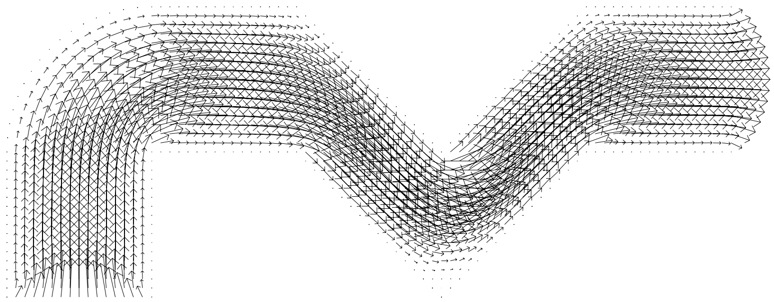

Figure 10: Solution of pipe-flow problem

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