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Non-conforming Nodal-value Discretization and Multigrid Methods for 4^{th}-order Problems

Boundary value problems for 4^{th} order PDE's arising from plate and shell models are considered. Because C^1-continuous finite elements are very expensive, usually non-conforming elements are used, e.g. Morley's triangle. The known non-conforming elements need Hermite-type degrees of freedom. To build elements using just Lagrange-type degrees of freedom, we suggest to use nodes outside of the element. Existing codes for second order equations can be extended very easily to this discretization.
The linear system is solved by a multigrid method. Because only nodal values are used, the grid transfer operations are very simple. Numerical examples verify the convergence rate of the discretization as well as the multigrid rate.

1. Kirchhoff Plate Model

On $V = H_0^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ we consider the variational problem find $u \in V : a(u,v) = \langle f, v\rangle \ \forall v \in V$, with the symmetric, elliptic and continuous bilinear-form a and the continuous linear-form f

$$a(u,v) = \int_\Omega \left[\sigma \Delta u \Delta v + (1-\sigma)\nabla^2 u : \nabla^2 v\right] \mathrm{dx}, \qquad \langle f, v\rangle = \int_\Omega f v \, \mathrm{dx}, \tag{1}$$

where $\sigma \in (0, 0.5)$. If Ω is convex and polygonal, then $u \in H^3(\Omega)$ and $\|u\|_3 \le c\,\|f\|_{-1}$ (see [2]).

2. Discretization

First, we introduce the discretization on the rectangle. The domain is dissected into triangles $\{T_i\}$, where each triangle contains 6 nodes as shown in Fig. 1. The restriction of the finite element function onto T_i is the quadratic polynomial, which extension to $\mathbb{R}^2$ interpolates in the 6 nodes. The reference element with the 6 Lagrangian degrees of freedom is shown in Fig. 2. The calculation of the derivatives ∇u_h in the mid-points of the edges give

$$\nabla u_h(a) = \begin{pmatrix} \frac{u_2+u_4}{2} - \frac{u_1+u_3}{2} \\ \frac{u_3+u_4}{2} - \frac{u_1+u_2}{2} \end{pmatrix} \quad \nabla u_h(b) = \begin{pmatrix} \frac{u_2+u_3}{2} - \frac{u_1+u_5}{2} \\ u_3 - u_1 \end{pmatrix} \quad \nabla u_h(c) = \begin{pmatrix} u_2 - u_1 \\ \frac{u_2+u_3}{2} - \frac{u_1+u_6}{2} \end{pmatrix}.$$

These derivatives coincide with the corresponding ones in the neighbour elements. Therefore the finite element space V_h is a (true) sub-space of the space V_h^M generated by the Morley - element [6].

On $V_h \not\subset H_0^2$ we define the norm $\|v_h\|_{2,h}^2 := \sum_{T_i} \|v_h\|_{2,T_i}^2$, the elliptic and continuous bilinear-form $a_h(u_h, v_h) = \sum_{T_i} \int_{T_i} [...] \mathrm{dx}$ and the continuous linearform $\langle f_h, v_h\rangle := \langle f, v_h^I\rangle$ with the linear interpolant v_h^I. Let $u_h \in V_h$ solve $a_h(u_h, v_h) = \langle f_h, v_h\rangle \ \forall v_h \in V_h$, and let u_h^M solve on V_h^M. The estimate $\|u - u_h^M\|_{2,h} \le ch\|u\|_3$ is show in [1].

Theorem 1. *The discretization error fulfills*

$$\|u - u_h\|_{2,h} \le ch\|u\|_3 \tag{2}$$

Proof. By the second lemma of Strang. Approximation is estimated by interpolation and element-wise extension to the enlarged triangle $\widetilde{T}_i$

$$\inf_{v_h \in V_h} \|u - v_h\|_{2,h}^2 \le \sum_{T_i \in \mathcal{T}_h} \|u - I_h u\|_{2,T_i}^2 \le \sum_{T_i \in \mathcal{T}_h} \|u - \widetilde{I_h^i u}\|_{2,\widetilde{T}_i \cap \Omega}^2 \le c \sum_{T_i \in \mathcal{T}_h} h^2 |u|_{3,\widetilde{T}_i \cap \Omega}^2 \le 4ch^2 |u|_3^2.$$

Consistency follows from Morley convergence

$$\sup_{w_h \in V_h} \frac{|a_h(u, w_h) - \langle f_h, w_h\rangle\,|}{\|w_h\|_{2,h}} = \sup_{w_h \in V_h} \frac{|a_h(u - u_h^M, w_h)|}{\|w_h\|_{2,h}} \le \|a_h\|\,\|u - u_h^M\|_{2,h} \le ch\|u\|_3.$$

A simple possibility to implement essential boundary conditions is by penalty terms. Assume edge 1-2 of the element in Fig. 2 is part of the Dirichlet boundary. The condition $u = 0$ is approximated by adding the large value ε^{-1} to the diagonal elements corresponding to nodes 1 and 2, while $\partial_n u = 0$ is reduced to edge mid-points and approximated by adding $\varepsilon^{-1}(u_3 + u_2 - u_1 - u_6)^2$ to the bilinear-form.

The suggested element cannot be used directly on unstructured meshes. We assume to have a coarse macro triangulation, which is hierarchically refined several times. On the interfaces, where the uniform structure is violated, normal derivative dofs are added in the mid-points of the edges (label a in Fig. 3). Then the inclusion $V_h \subset V_h^M$ is still valid. The realization can be done by adding 2 nodes (nodes 4 and 6) on the uniform mesh and enforce continuity of $\partial_n u$ again by penalty.

To overcome the restriction to polygonal domains, at least two approaches are possible. Either a smooth function mapping the macro element to a curvilinear triangle is included into the bilinearform, or the nodes of the macro element are mapped to the curvilinear domain, where the original bilinear-form is used. The deviation of the uniform mesh is of lower order and therefore the rate of convergence is conserved.

3. Multigrid Solver and Numerical Result

Multigrid methods for 4^{th} order problems are presented in [4], [5], [7], an abstract framework for non-conforming problems is developed in [3]. The theory from [4] can be adapted to this discretization scheme. But numerical experiments show, that also the simpler, linear prolongation gives optimal convergence rates. We confirm the results of [5], that the W-cycle can diverge, the V-cycle is not optimal, but the generalized V-cycle leads to optimal rates. To handle non-pointwise penalty terms, local problems as shown in Fig. 4 are solved in the block-GS smoother. Results for a problem on a circle decomposed into 4 macro elements using smooth nodal displacement are given in the table beside.

N	$\|u_h - I_h u\|_{2,h}$	cg-its
125	0.21	9
325	0.086	13
917	0.032	16
2869	0.012	17
9845	0.0041	19
36085	0.0016	20
137717	0.0006	20

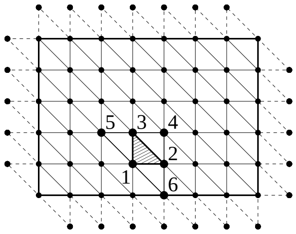

Fig 1: Discretization

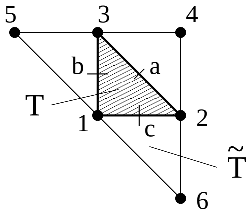

Fig. 2: Reference element

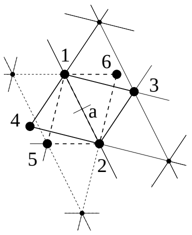

Fig. 3: Interface

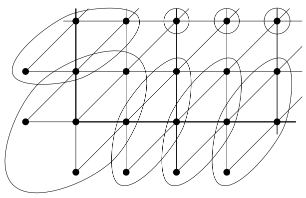

Fig. 4: Smoother blocks

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