Multigrid Methods for a Class of Parameter Dependent Problem in Primal Variables

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Abstract

This paper deals with the construction and analysis of multigrid methods for parameter dependent problems. The considered algorithms can be applied equivalently for the positive definite system in primal variables, or for the saddle point system. We formulate assumptions, on which the multigrid analysis based. Finally, we verify these assumptions for specific examples including nearly incompressible materials and the Reissner Mindlin plate model.

1 Introduction

In this paper we consider a class of problems depending in a certain way on a small parameter ε. Several such examples stem from computational mechanics, where material parameters (e.g. the Poisson ration) may cause difficulties, or the shape of the body degenerates and lead to a model of structural mechanics.
A straight forward finite element discretization of these problem gives bad results. This is called locking effect, [3], [1]. One way out is the reformulation as a mixed variational problem, where the finite element discretization is done [12]. Often it is possible to retain to a positive definite problem in the primal variables, what is advantageously for practical computations.
Like the discretization error, also the convergence rate of iterative methods like multigrid with standard components degenerates. So robust multigrid methods for the indefinite linear system have been designed, see [4], [10], [5] and the references therein. In these papers the norms stemming from the saddle point system are used. The approximation property is essentially the same, but the type of smoother varies. The technique from [5]

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is 'u - dominated', so it can also be applied in primal variables [21]. In [23] the smoothing property for the approximative version of the Braess-Sarazin smoother is verified.
The present paper is an extension to [19], where the case of nearly incompressible materials was considered. Here, we formulate first conditions to a generic problem and prove multigrid convergence. Then the conditions are checked for the case of nearly incompressible materials with conforming or non-conforming discretization and for the Reissner Mindlin plate model with the finite element discretization from [13]. The limit problems with $\varepsilon = 0$, e.g. Stokes, can be handled in the mixed form.
The key features of the algorithm are stable grid transfer operators and block smoothers capturing base functions of the kernel of the up-scaled term. The analysis is based on norms related to the primal problem. The philosophy is the following. For the primal variable, the duality technique can be applied to achieve better convergence in a weaker norm. The dual variable is decomposed orthogonally into the coarse grid part and the local part. The grid transfer operators must take care of this decomposition. It seems to be interesting, that the considered problems need different decompositions of the dual variable.
Because some parts of the analysis are similar to [19], we skip here some proofs.
The rest of the paper is organized as follows. In Section 2 the problem formulation and discretization is given, Section 3 defines the necessary multigrid ingredients. In Sections 4, 5 and 6 multigrid methods are formulated and analyzed based only on the assumptions of the former sections. In the Section 7 we check the assumptions for nearly incompressible materials and Stokes, and in Section 8 we verify them for the Reissner Mindlin plate model. The author recommends to switch reading between Sections 2 and 3, and Sections 7 and 8.

2 Formulation and Discretization

We consider variational problems find $u \in V$ such that

$$A(u, v) = \langle f, v \rangle \qquad \forall\, v \in V \tag{1}$$

with the symmetric bilinear-form $A(.,.)$ and the linear functional f on the Hilbert space V. The bilinear-form can be written as

$$A(u, v) = a(u, v) + \varepsilon^{-1} (\Lambda u, \Lambda v)_0 \tag{2}$$

with the symmetric bilinear-form $a(.,.)$, the linear operator $\Lambda : V \to L_2$ with a non-trivial kernel and the small, positive parameter ε. We assume that $a(u, v) + (\Lambda u, \Lambda v)$ is elliptic and continuous on V. So $A(.,.)$ is it as well, but the ratio of the constants degenerate as ε becomes small.
By definition of

$$p = \varepsilon^{-1} \Lambda u \tag{3}$$

in $Q := L_2$ we obtain the equivalent mixed problem to find $(u,p) \in X := V \times Q$ such that

$$B((u,p),(v,q)) = \langle f, v \rangle \qquad \forall\, (v,q) \in X \tag{4}$$

with the block bilinear-form $B(.,.) : X \times X \to \mathbf{R}$ defined as

$$B((u,p),(v,q)) = a(u,v) + (\Lambda u, q)_0 + (\Lambda v, p)_0 - \varepsilon\, (p,q)_0. \tag{5}$$

We will need the stronger norms $\|.\|_{V^2}$ and $\|.\|_{Q^1}$ on dense sub-spaces $V^2 \subset V$ and $Q^1 \subset Q$ and the weaker norm $\|.\|_{V^0}$ on the larger space $V^0 \supset V$. For the Stokes problem the indices coincide with the according Sobolev space indices. If the linear functional f is in the more regular space $(V^0)^*$, the solution should be more regular as well and should be bounded by

$$\|u\|_{V^2} + \|p\|_{Q^1} \preceq \|f\|_{(V^0)^*}. \tag{6}$$

The symbol $a \preceq b$ does mean there exists a constant c independent of a and b, as well as ε and the level of refinement l, such that $a \leq c\, b$. We write $a \succeq b$ for $b \preceq a$, and $a \simeq b$ for $a \preceq b$ and $b \preceq a$.
The discretization of (4) is based on a sequence of possibly non-nested non-conforming finite element spaces

$$V_1, V_2, \ldots, V_L$$

and nested spaces

$$Q_1 \subset Q_2 \subset \ldots Q_L \subset Q$$

with norms $\|.\|_{V,l}$ and $\|.\|_{Q,l}$. We set $X_l = V_l \times Q_l$ with the product norm $\|(u,p)\|_{X,l} := (\|u\|^2_{V,l} + \|p\|^2_{Q,l})^{1/2}$. The index l denotes the level and usually runs from 1 to L. On each level we use the symmetric bilinear-form

$$B_l((u_l,p_l),(v_l,q_l)) = a_l(u_l,v_l) + b_l(u_l,q_l) + b_l(v_l,p_l) - \varepsilon\, c_l(p_l,q_l) \tag{7}$$

with positive bilinear-form $c_l(.,.)$ defining the norm $\|.\|_{c_l}$ and elliptic $a_l(.,.)$

$$a_l(u_l,u_l) \succeq \|u_l\|^2_{V,l} \qquad \forall\, u_l \in V_l. \tag{8}$$

It is assumed to be continuous

$$B_l((u_l,p_l),(v_l,q_l)) \preceq \|(u_l,p_l)\|_{X,l}\, \|(v_l,q_l)\|_{X,l} \tag{9}$$

and stable

$$\sup_{(v_l,q_l)\in X_l} \frac{B_l((u_l,p_l),(v_l,q_l))}{\|(v_l,q_l)\|_{X,l}} \succeq \|(u_l,p_l)\|_{X,l} \qquad \forall\, (u_l,p_l) \in X_l. \tag{10}$$

On the spaces V_l the inverse estimate

$$\|u_l\|_{V,l} \preceq h_l^{-1} \|u_l\|_{V^0} \tag{11}$$

should be fulfilled with coefficients h_l. We think on the mesh-size on multigrid level l.

We require interpolation operators

$$I_l^X = (I_l^V, I_l^Q) : (V^2 \times Q^1) \to X_l \tag{12}$$

fulfilling

$$h_l^{-1}\|v - I_l^V v\|_{V^0} + \|(v,q) - I_l^X(v,q)\|_{X,l} \preceq h_l \left(\|v\|_{V^2} + \|q\|_{Q^1}\right) \tag{13}$$

for given $(v,q) \in V^2 \times Q^1$.
Let (u,p) be the solution of (4) and let $(u_l, p_l) \in X_l$ be the solution of the mixed finite element problem

$$B_l((u_l,p_l),(v_l,q_l)) = \langle f, v_l\rangle \qquad \forall\,(v_l,q_l) \in X_l. \tag{14}$$

Then the a priori estimate

$$h^{-1}\|u - u_l\|_{V^0} + \|u - u_l\|_{V,l} + \|p - p_l\|_{Q,l} \preceq h\,\|f\|_{(V^0)^*} \tag{15}$$

should be valid.
By the definiteness of $c_l(.,.)$ on the finite dimensional spaces Q_l we can define the operators

$$\Lambda_l : V_l \to Q_l : \qquad c_l(\Lambda_l u_l, q_l) = b_l(u_l, q_l) \qquad \forall\, u_l \in V_l,\ \forall\, q_l \in Q_l. \tag{16}$$

It is of special interest, if Λ_l can be computed locally. Then we can assemble the reduced forms

$$A_l(u_l, v_l) := a_l(u_l, v_l) + \varepsilon^{-1} c_l(\Lambda_l u_l, \Lambda_l v_l) \tag{17}$$

The positive definite finite element problem in primal variables to find $u_l \in V_l$ such that

$$A_l(u_l, v_l) = \langle f, v_l\rangle \qquad \forall\, v_l \in V_l \tag{18}$$

provides the same solution u_l as (14).

3 Multigrid ingredients

Due to the non-nestedness of the spaces we have to define prolongation operators $R_l^{V,0} : V_{l-1} \to V_l$, with $2 \le l \le L$. They are called trivial prolongation operators and should fulfill the continuity and approximation estimates

$$\begin{aligned} \|R_l^{V,0} u_{l-1}\|_{V,l} + h^{-1}\,\|u_{l-1} - R_l^{V,0} u_{l-1}\|_{V^0} &\preceq \|u_{l-1}\|_{V,l-1}, \\ \|I_l u - R_l^{V,0} I_{l-1} u\|_{V^0} &\preceq h^2\,\|u\|_{V^2} \end{aligned} \tag{19}$$

for all $u \in V^2$ and $u_{l-1} \in V_{l-1}$. They are extended to $R_l^{X,0} = (R_l^{V,0}, id) : X_{l-1} \to X_l$. We assume norm equivalence across two levels

$$\|p_{l-1}\|_{Q,l-1} \simeq \|p_{l-1}\|_{Q,l} \qquad \forall\, p_{l-1} \in Q_{l-1}. \tag{20}$$

The space Q_l is $c_l(.,.)$-orthogonal decomposed as

$$Q_l = Q_{l,T} + \widetilde{Q}_{l-1} \tag{21}$$

with $\widetilde{Q}_{l-1} \subset Q_{l-1}$. Let $I_{l-1}^{\widetilde{Q}}$ denote the projector $Q_l \to \widetilde{Q}_{l-1}$.
Combine $Q_{l,T}$ with $V_{l,T} \subset V_l$ to form

$$X_{l,T} = V_{l,T} \times Q_{l,T}. \tag{22}$$

It should split $B(.,.)$-orthogonal into local spaces of small degree. The space $X_{l,T}$ should be stable such that for all $(u,p) \in X_{l,T}$

$$\sup_{(v,q)\in X_{l,T}} \frac{B_l((u,p),(v,q))}{\|(v,q)\|_{X,l}} \succeq \|(u,p)\|_{X,l} \tag{23}$$

is fulfilled, and the inclusion

$$\Lambda_l V_{l,T} \subset Q_{l,T} \tag{24}$$

is assumed. The trivial prolongation is such that

$$B_{l-1}((u_{l-1},p_{l-1}),(0,\tilde{q}_{l-1})) = B_l(R_l^{X,0}(u_{l-1},p_{l-1}),(0,\tilde{q}_{l-1})) \tag{25}$$

holds for all $(u_{l-1},p_{l-1}) \in X_{l-1}$ and for all $q_{l-1} \in \widetilde{Q}_{l-1}$. On the space $V_{l,T}$ not only the inverse estimate (11), but also the norm equivalence

$$\|u_{l,T}\|_{V^0} \simeq h\,\|u_{l,T}\|_{V,l} \qquad \forall\, u_{l,T} \in V_{l,T} \tag{26}$$

is fulfilled.
For the construction of the smoother we need the eventually overlapping space decomposition

$$V_l = \sum_{i=1}^{N_i} V_{l,i}, \qquad Q_l = \sum_{i=1}^{N_i} Q_{l,i}. \tag{27}$$

We set $X_{l,i} = V_{l,i} \times Q_{l,i}$ and demand stability, namely for all $(u,p) \in X_{l,i}$ there holds

$$\sup_{(v,q)\in X_{l,i}} \frac{B_l((u,p),(v,q))}{\|(v,q)\|_{X,l}} \succeq \|(u,p)\|_{X,l}. \tag{28}$$

Additionally, the inclusion

$$\Lambda_l V_{l,i} \subset Q_{l,i} \tag{29}$$

must be fulfilled.
Based on the energy norm $\|u_l\|_{A_l} := A_l(u_l,u_l)^{1/2}$, we define the splitting norm

$$\|u_l\|_{D_l}^2 := \inf_{\substack{u_l=\sum u_{l,i} \\ u_{l,i}\in V_{l,i}}} \|u_{l,i}\|_{A_l}^2. \tag{30}$$

The norm for which we will prove the approximation property is

$$\|u_l\|_{V,l,\tilde{0}}^2 := h_l^{-2}\|u\|_{V^0}^2 + \varepsilon^{-1}\|\Lambda_l u_l\|_{c_l}^2 + \varepsilon^{-2}\|I_{l-1}^{\widetilde{Q}}\Lambda_l u_l\|_{Q,l}^2. \tag{31}$$

Assume that some interpolation norm ([17], see also [9]) with index $\alpha \in (0,1]$ can be estimated by

$$\|u_l\|_{[A_l,D_l]_\alpha}^2 \preceq \|u_l\|_{V,l,\tilde{0}}^2. \tag{32}$$

These are the ingredients we will use in the following three sections to formulate and analyze multigrid methods, and which we will verify for the considered problems.

4 Multigrid Formulation

In this section we formulate the multigrid method which will be analyzed in the following sections. Therefore, let $\langle .,. \rangle_l$ be an additional inner product on V_l. One might think of the Euclidean inner product of nodal values. We can define the linear operator (the system matrix) $A_l : V_l \to V_l$ implicitly by

$$\langle A_l u_l, v_l \rangle_l = A_l(u_l, v_l) \qquad \forall\, u_l, v_l \in V_l,$$

and the right hand side vector $f_l \in V_l$ by

$$\langle f_l, v_l \rangle_l = \langle f, v \rangle \qquad \forall\, v_l \in V_l.$$

So the variational problem is equivalent to the linear equation to find $u_l \in V_l$ such that

$$A_l u_l = f_l. \tag{33}$$

As smoother $S_l^A : V_l \to V_l$ we use the additive Schwarz method with the space splitting $V_{l,i}$ from (27). Let us define the projection

$$P_{l,i}^A : V_l \to V_{l,i} : \quad A_l(P_{l,i}^A u_l, v_{l,i}) = A_l(u_l, v_{l,i}) \qquad \forall\, u_l \in V_l \;\; \forall\, v_{l,i} \in V_{l,i}. \tag{34}$$

and the smoother as

$$S_l^A := I - \tau \sum_{i=1}^{N_l} P_{l,i}^A, \tag{35}$$

where τ is chosen such that S_l^A has a positive spectrum. We mention, that τ only depends on the number of overlapping sub-spaces. Recall the splitting norm $\|.\|_{D_l}$ from (30). This norm is a Hilbert space norm, thus it provides also the quadratic form $D_l(.,.)$, and the linear operator $D_l : V_l \to V_l$ defined implicitly by

$$\langle D_l u_l, v_l \rangle_l = D_l(u_l, v_l) \qquad \forall\, u_l, v_l \in V_l.$$

The additive Schwarz theory [22], [18] allows the explicit representation of D_l by

$$D_l^{-1} = \sum P_{l,i}^A A_l^{-1}.$$

Thus, the smoother can be rewritten as

$$S_l^A = I - \tau D_l^{-1} A_l.$$

The other multigrid component is the grid transfer. Therefore, we define the projections

$$P_{l,T}^A : V_l \to V_{l,T} : \quad A_l(P_{l,T}^A u_l, v_{l,T}) = A_l(u_l, v_{l,T}) \qquad \forall\, u_l \in V_l \;\; \forall\, v_{l,T} \in V_{l,T} \tag{36}$$

onto the sub-spaces $V_{l,T}$ defined in (22). The used prolongation operator is

$$R_l^V := (I - P_{l,T}^A) R_l^{V,0}. \tag{37}$$

Procedure $MG(l, u_l, f_l)$
if $l = 1$

$$MG(l, u_l, f_l) = A_l^{-1} f_l$$

else

$$\begin{aligned}
&u_l^{1,0} = u_l \\
&\text{do } j = 1, \ldots, m_l \\
&\qquad u_l^{1,j} = u_l^{1,j-1} + \tau D_l^{-1}(f_l - A_l u_l^{1,j-1}) \\
&d_{l-1} = [R_l^V]^T (f_l - A_l u_l^{1,m_l}) \\
&u_{l-1}^{2,0} = 0 \\
&\text{do } j = 1, \ldots, q \\
&\qquad u_{l-1}^{2,j} = MG(l-1, u_{l-1}^{2,j-1}, d_{l-1}) \\
&u_l^{3,0} = u_l^{1,m_l} + R_l^V u_{l-1}^{2,q} \\
&\text{do } j = 1, \ldots, m_l \\
&\qquad u_l^{3,j} = u_l^{3,j-1} + \tau D_l^{-1}(f_l - A_l u_l^{3,j-1}) \\
&MG(l, u_l, f_l) = u_l^{3,m_l}
\end{aligned}$$

Figure 1: Multigrid Algorithm

We mention, that the projection $P_{l,T}^A$ can be efficiently computed, because $V_{l,T}$ splits also A_l - orthogonal into sub-spaces of small dimension. The $\langle ., . \rangle_l$ - $\langle ., . \rangle_{l-1}$ adjoint restriction operator is

$$[R_l^V]^T := [R_l^{V,0}]^T (I - P_{l,T}^A)^T \tag{38}$$

with the adjoint trivial restriction operator $[R_l^{V,0}]^T : V_l \to V_{l-1}$ defined as

$$\left\langle [R_l^{V,0}]^T u_l, v_{l-1} \right\rangle_{l-1} = \left\langle u_l, R_l^{V,0} v_{l-1} \right\rangle_l \qquad \forall\, u_l \in V_l \;\; \forall\, v_{l-1} \in V_{l-1},$$

and $(I - P_{l,T}^A)^T = A_l (I - P_{l,T}^A) A_l^{-1}$ is the $\langle ., . \rangle_l$ adjoint of $(I - P_{l,T}^A)$.
One step of the multigrid iteration

$$\hat{u}_l = MG(l, u_l, f_l) \tag{39}$$

is defined in Algorithm 1.
It leads to the multigrid operator $M_l^A : V_l \to V_l$

$$M_l^A u_l := MG(l, u_l, 0) \tag{40}$$

and to the multigrid preconditioner $C_l : V_l \to V_l$ defined implicitly as

$$C_l^{-1} f_l := MG(l, 0, f_l). \tag{41}$$

The multigrid operator does fulfill the recursion formula

$$\begin{aligned}
M_1^A &= 0, \\
M_l^A &= (S_l^A)^{m_l} \left(I - R_l^V (I - (M_{l-1}^A)^q) A_{l-1}^{-1} [R_l^V]^T A_l \right) (S_l^A)^{m_l}.
\end{aligned} \tag{42}$$

Next, we define a multigrid method for the mixed formulation. Let B_l be defined by an inner product on X_l. The projections $P^B_{l,T} : X_l \to X_{l,T}$ and $P^B_{l,i} : X_l \to X_{l,i}$ are defined by

$$B_l(P^B_{l,T}(u_l, p_l), (v_{l,T}, q_{l,T})) = B_l((u_l, p_l), (v_{l,T}, q_{l,T})) \quad \forall\, (v_{l,T}, q_{l,T}) \in X_{l,T} \tag{43}$$

and

$$B_l(P^B_{l,i}(u_l, p_l), (v_{l,i}, q_{l,i})) = B_l((u_l, p_l), (v_{l,i}, q_{l,i})) \quad \forall\, (v_{l,i}, q_{l,i}) \in X_{l,i} \tag{44}$$

for all $(u_l, p_l) \in X_l$. Then the smoother and the prolongation operator are set to

$$\begin{aligned} S^B_l &= I - \tau \sum P^B_{l,i}, \\ R^X_l &= (I - P^B_{l,T}) R^{X,0}_l. \end{aligned}$$

Applying the same multigrid algorithm 1 we obtain the saddle point multigrid operator

$$\begin{aligned} M^B_1 &= 0, \\ M^B_l &= (S^B_l)^{m_l} \left(I - R^X_l (I - (M^B_{l-1})^q) B^{-1}_{l-1} [R^X_l]^T B_l \right) (S^B_l)^{m_l}. \end{aligned} \tag{45}$$

5 Multigrid Analysis

First, we collect some observations, which proofs are simple generalizations to that given in [19] and thus omitted. For the analysis, the sub-spaces $X_{l,0} \subset X_l$

$$X_{l,0} = \{(u_l, p_l) \in X_l : \Lambda_l u_l = \varepsilon p_l\} \tag{46}$$

plays an essential role. For all $(u_l, p_l) \in X_{l,0}$ there holds

$$B_l((u_l, p_l), (0, q_l)) = 0 \qquad \forall\, q_l \in Q_l \tag{47}$$

as well as

$$A_l(u_l, v_l) = B_l((u_l, p_l), (v_l, 0)) \qquad \forall\, v_l \in V_l. \tag{48}$$

We define the mixed local norm

$$\|(u_l, p_l)\|^2_{X,l,\tilde{0}} := h_l^{-2} \|u_l\|^2_{V^0} + \varepsilon \, \|p_l\|^2_{c_l} + \|I^{\widetilde{Q}}_{l-1} p_l\|^2_{Q,l} \tag{49}$$

and the mixed energy norm

$$\|(u_l, p_l)\|^2_{B_l} := a_l(u_l, u_l) + \varepsilon c_l(p_l, p_l). \tag{50}$$

On the space $X_{l,0}$ they reduce to the primal norms $\|u_l\|_{V,l,\tilde{0}}$ defined in (31), and the energy norm $\|u_l\|_{A_l}$, respectively.
The defined mixed multigrid components preserve the sub-space $X_{l,0}$:

Lemma 1

1. *The projections $P^B_{l,i}$ and $P^B_{l,T}$ are well defined and uniformly bounded on X_l with respect to the $\|.\|_{X,l}$-norm.*

2. *The projections $P^B_{l,i}$ and $P^B_{l,T}$ map $X_{l,0}$ into itself, and they are bounded by 1 with respect to the $\|.\|_{B_l}$-norm on $X_{l,0}$.*

3. *The prolongation operator R^X_l maps $X_{l-1,0}$ into X_l, and the coarse grid solution operator $B^{-1}_{l-1}[R^X_l]^T B_l$ maps $X_{l,0}$ into $X_{l-1,0}$.*

On the sub-space $X_{l,0}$ the mixed components can be reduced to the primal once:

Lemma 2

1. *Let $(u_l, p_l) \in X_{l,0}$ and $(\hat{u}_l, \hat{p}_l) = P^B_{l,i}(u_l, p_l)$. Then $\hat{u}_l = P^A_{l,i} u_l$.*

2. *Let $(u_{l-1}, p_{l-1}) \in X_{l-1,0}$ and $(\hat{u}_l, \hat{p}_l) = R^X_l (u_l, p_l)$. Then $\hat{u}_l = R^V_l u_{l-1}$.*

3. *Let $(u_l, p_l) \in X_{l,0}$ and $(\hat{u}_{l-1}, \hat{p}_{l-1}) = B^{-1}_{l-1}[R^X_l]^T B_l (u_l, p_l)$. Then $\hat{u}_{l-1} = A^{-1}_{l-1}[R^V_l]^T A_l u_l$.*

By induction on l we obtain

Theorem 1 (Equivalence of algorithms)
Let $(u_l, p_l) \in X_{l,0}$ and $(\hat{u}_l, \hat{p}_l) = M^B_l (u_l, p_l)$. Then $\hat{u}_l = M^A_l u_l$ and $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$.

The multigrid analysis is based on the estimate of the two-grid operators, the primal

$$\hat{M}^A_l = (S^A_l)^{m_l} \left(I - R^V_l A^{-1}_{l-1} [R^V_l]^T A_l \right) (S^A_l)^{m_l}, \tag{51}$$

and the corresponding mixed $\hat{M}^B_l$. We will verify the approximation property and the smoothing property to obtain the two-grid convergence estimate

Theorem 2 (Two-Grid Convergence)
The two-grid operator $\hat{M}^A_l$ can be estimated by

$$\|\hat{M}^A_l\|_{A_l} \leq c\, m_l^{-\alpha/2}, \tag{52}$$

with a constant c independent of l and ε. The two-grid operator $\hat{M}^B_l$ maps $X_{l,0}$ into itself and is bounded on $X_{l,0}$ by

$$\|\hat{M}^B_l\|_{B_l} \leq c\, m_l^{-\alpha/2} \tag{53}$$

with a constant c independent of l and ε. The power α is the the index of the interpolation estimate (32).

Proof: Define for $(u^0, p^0) \in X_{l,0}$

$$(u^1, p^1) = \left(I - R^X_l B^{-1}_{l-1} [R^X_l]^T B_l \right) (u^0, p^0) \tag{54}$$

and

$$(u^2, p^2) = (S^B_l)^{m_l} (u^1, p^1). \tag{55}$$

By the previous lemmata we get $(u^1, p^1) \in X_{l,0}$ and

$$u^1 = \left(I - R^V_l A^{-1}_{l-1} [R^V_l]^T A_l \right) u^0$$

In Section 6 we will prove the approximation property (see Theorem 4)

$$\|(u^1,p^1)\|_{X,l,\tilde{0}} = \|u^1\|_{V,l,\tilde{0}} \preceq \|u^0\|_{A_l} = \|(u^0,p^0)\|_{B_l} \tag{56}$$

using the mixed form. We also get $(u^2,p^2) \in X_{l,0}$ and

$$u^2 = (S_l^A)^{m_l} u^1.$$

One half of the smoothing property is the algebraic part

$$\|(S_l^A)^m u^1\|_{A_l} \preceq m^{-\alpha/2} \|u^1\|_{[A_l,D_l]_\alpha},$$

which follows from

$$\|(S_l^A)^m u_l\|_{A_l} \preceq \|u_l\|_{A_l} \qquad \forall\, u_l \in V_l$$

and

$$\|(S_l^A)^m u_l\|_{A_l} \preceq m^{-1/2} \|u_l\|_{D_l} \qquad \forall\, u_l \in V_l$$

by the use of operator interpolation with index $\alpha \in [0,1]$. We mention, that the constants hidden in $\preceq$ do not depend on m. The combination with the problem specific part stated in (32) gives

$$\|(S_l^A)^m u^1\|_{A_l} \preceq m^{-\alpha/2} \|u^1\|_{V,l,\tilde{0}}.$$

Combining both properties proves the theorem. □

The following theorem follows by standard techniques [16], [8], [7].

Theorem 3 (Multigrid Convergence)

- *The norm of the W-cycle operator is bounded independently of L and ε if the number of smoothing steps m_l is sufficiently large.*
- *The variable V-cycle operator with $m^l = 2^{L-l}$ leads to a preconditioner C_L with condition number $\kappa(C_L^{-1}A_L)$ bounded independently of L and ε.*

6 Approximation Property

The coarse grid correction operator

$$I - R_l^X B_{l-1}^{-1} [R_l^X]^T B_l = I - (I - P_{l,T}^B) R_l^{X,0} B_{l-1}^{-1} [R_l^{X,0}]^T B_l (I - P_{l,T}^B)$$

mapping $(u_1,p_1) \in X_{l,0} \to (u_5,p_5) \in X_{l,0}$ is split into

$$\begin{aligned}
(u_2,p_2) &= (I - P_{l,T}^B)(u_1,p_1), \\
(u_3,p_3) &= B_{l-1}^{-1} [R_l^{X,0}]^T B_l (u_2,p_2), \\
(u_4,p_4) &= (I - P_{l,T}^B) R_l^{X,0} (u_3,p_3), \\
(u_5,p_5) &= (u_1,p_1) - (u_4,p_4).
\end{aligned} \tag{57}$$

Theorem 4 (Approximation Property)
Let $(u_1, p_1) \in X_{l,0}$ and compute (u_5, p_5) by (57). Then the approximation property

$$\|(u_5, p_5)\|_{X,l,\tilde{0}} \preceq \|(u_1, p_1)\|_{B_l} \tag{58}$$

is valid.

Proof: We use the triangle inequality and the three lemmata proven below to obtain the result

$$\begin{aligned}
\|(u_5, p_5)\|_{X,l,\tilde{0}} &= \|(u_1, p_1) - (u_4, p_4)\|_{X,l,\tilde{0}} \\
&\leq \|(u_1, p_1) - (u_2, p_2)\|_{X,l,\tilde{0}} + \|(u_2, p_2) - R_l^{X,0}(u_3, p_3)\|_{X,l,\tilde{0}} + \\
&\quad \|R_l^{X,0}(u_3, p_3) - (u_4, p_4)\|_{X,l,\tilde{0}} \\
&\preceq \|(u_1, p_1)\|_{B_l}.
\end{aligned}$$

□

Lemma 3
With the notation of (57) there holds

$$\|(u_2, p_2)\|_{B_l} + \|(u_2, p_2) - (u_1, p_1)\|_{X,l,\tilde{0}} + \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}. \tag{59}$$

Proof: Lemma 1 gives $\|P_{l,T}^B(u_1, p_1)\|_{B_l} \preceq \|(u_1, p_1)\|_{B_l}$, which bounds the first term. To bound the second term we use $(u_2, p_2) - (u_1, p_1) \in X_{l,T}$, the norm equivalence (26), the orthogonal decomposition (21), and the ellipticity (8) of a_l

$$\begin{aligned}
\|(u_2, p_2) - (u_1, p_1)\|_{X,l,\tilde{0}}^2 &= h_l^{-2}\|u_2 - u_1\|_{V^0}^2 + \varepsilon \|p_2 - p_1\|_{c_l}^2 + \|I_{l-1}^{\widetilde{Q}}(p_2 - p_1)\|_{Q,l}^2 \\
&\preceq \|u_2 - u_1\|_{V,l}^2 + \varepsilon \|p_2 - p_1\|_{c_l}^2 \preceq \|(u_2 - u_1, p_2 - p_1)\|_{B_l}^2 \preceq \|(u_1, p_1)\|_{B_l}^2.
\end{aligned}$$

From $p_2 - I_{l-1}^{\widetilde{Q}} p_2 \in Q_{l,T}$, stability (23) of $X_{l,T}$, inclusion (24) and orthogonality (21) and the definition of $P_{l,T}^B$ we obtain

$$\begin{aligned}
\|p_2 - I_{l-1}^{Q} p_2\|_{Q,l} &\preceq \sup_{(v,q)\in X_{l,T}} \frac{B_l((0, p_2 - I_{l-1}^{\widetilde{Q}} p_2), (v, q))}{\|(v, q)\|_{X,l}} \\
&= \sup_{(v,q)\in X_{l,T}} \frac{B_l((0, p_2), (v, q)) - c_l(I_{l-1}^{\widetilde{Q}} p_2, \Lambda_l v - q)}{\|(v, q)\|_{X,l}} = \sup_{(v,q)\in X_{l,T}} \frac{B_l((0, p_2), (v, q))}{\|(v, q)\|_{X,l}} \\
&= \sup_{(v,q)\in X_{l,T}} \frac{B_l((-u_2, 0), (v, q))}{\|(v, q)\|_{X,l}} \preceq \|u_2\|_{V,l} \preceq \|(u_2, p_2)\|_{B_l}^2.
\end{aligned}$$

□

Lemma 4
With the notation of (57) there holds

$$\|(u_3, p_3)\|_{B_{l-1}} + \|R_l^{X,0}(u_3, p_3) - (u_2, p_2)\|_{X,l,\tilde{0}} + \|p_3 - I_{l-1}^{\widetilde{Q}} p_3\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}. \tag{60}$$

Proof: From the variational definition of (u_3, p_3)

$$B_{l-1}((u_3,p_3),(v,q)) = B_l((u_2,p_2),R_l^{X,0}(v,q)) \qquad \forall\,(v,q)\in X_{l-1}$$

and property (25) we get

$$B_{l-1}((u_3,p_3-I_{l-1}^{\widetilde{Q}}p_2),(v,q)) = B_l((u_2,p_2-I_{l-1}^{\widetilde{Q}}p_2),R_l^{X,0}(v,q)) \qquad \forall\,(v,q)\in X_{l-1}.$$

Stability on X_{l-1}, continuity of $B_l(.,.)$ and the prolongation operator $R_l^{X,0}$, and Lemma 3 gives

$$\begin{aligned}
\|u_3\|_{V,l-1} + \|p_3 - I_{l-1}^{\widetilde{Q}}p_2\|_{Q,l-1} &\preceq \sup_{(v,q)\in X_{l-1}} \frac{B_{l-1}((u_3,p_3-I_{l-1}^{\widetilde{Q}}p_2),(v,q))}{\|(v,q)\|_{X,l-1}} \\
&= \sup_{(v,q)\in X_{l-1}} \frac{B_l((u_2,p_2-I_{l-1}^{\widetilde{Q}}p_2),R_l^{X,0}(v,q))}{\|(v,q)\|_{X,l-1}} \\
&\preceq \|u_2\|_{V,l} + \|p_2 - I_{l-1}^{\widetilde{Q}}p_2\|_{Q,l} \preceq \|(u_1,p_1)\|_{B_l}.
\end{aligned}$$

Norm equivalence (20) on Q_{l-1} and Q_l, and Lemma 3 gives

$$\|p_3 - p_2\|_{Q,l} \preceq \|p_3 - I_{l-1}^{\widetilde{Q}}p_2\|_{Q,l-1} + \|p_2 - I_{l-1}^{\widetilde{Q}}p_2\|_{Q,l} \preceq \|(u_1,p_1)\|_{B_l},$$

the estimate

$$\|p_3 - I_{l-1}^{\widetilde{Q}}p_3\|_{Q,l} \le \|p_3 - p_2 + I_{l-1}^{\widetilde{Q}}(p_2-p_3)\|_{Q,l} + \|p_2 - I_{l-1}^{\widetilde{Q}}p_2\|_{Q,l} \preceq \|(u_1,p_1)\|_{B_l},$$

and with continuity of $B_l(.,.)$ also

$$\varepsilon\,\|p_3\|_{c_l}^2 \preceq \varepsilon\,\|p_2-p_3\|_{c_l}^2 + \varepsilon\,\|p_2\|_{c_l}^2 \preceq \|p_2-p_3\|_{Q,l}^2 + \varepsilon\,\|p_2\|_{c_l}^2 \preceq \|(u_1,p_1)\|_{B_l}^2.$$

The only term left is the improved estimate in the weaker norm, which we will bound as

$$\|u_2 - R_l^{V,0}u_3\|_{V^0} \preceq h_l\,\|(u_2,p_2-I_{l-1}^{\widetilde{Q}}p_2)\|_{X,l}. \tag{61}$$

Therefore, we define the dual problem

$$B((\varphi,\psi),(v,q)) = (u_2 - R_l^{V,0}u_3, v)_{V^0}.$$

The right hand side is a linear functional in $(V^0)^*$ with norm $\|u_2 - R_l^{V,0}u_3\|_{V^0}$. We also state the finite element problems

$$\begin{aligned}
B_l((\varphi_l,\psi_l),(v_l,q_l)) &= (u_2 - R_l^{V,0}u_3, v_l)_{V^0} && \forall\,(v_l,q_l)\in X_l, \\
B_{l-1}((\varphi_{l-1},\psi_{l-1}),(v_{l-1},q_{l-1})) &= (u_2 - R_l^{V,0}u_3, v_{l-1})_{V^0} && \forall\,(v_{l-1},q_{l-1})\in X_{l-1}, \\
B_{l-1}((\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1}),(v_{l-1},q_{l-1})) &= (u_2 - R_l^{V,0}u_3, R_l^{V,0}v_{l-1})_{V^0} && \forall\,(v_{l-1},q_{l-1})\in X_{l-1}.
\end{aligned}$$

First, we estimate the error due to the differences in the right hand side

$$
\begin{aligned}
&\|(\varphi_{l-1}-\widetilde{\varphi}_{l-1},\psi_{l-1}-\widetilde{\psi}_{l-1})\|_{X,l-1} \qquad (62)\\
&\quad\preceq \sup_{(v_{l-1},q_{l-1})\in X_{l-1}} \frac{B_{l-1}((\varphi_{l-1}-\widetilde{\varphi}_{l-1},\psi_{l-1}-\widetilde{\psi}_{l-1}),(v_{l-1},q_{l-1}))}{\|(v_{l-1},q_{l-1})\|_{X,l-1}}\\
&\quad= \sup_{(v_{l-1},q_{l-1})\in X_{l-1}} \frac{(u_2-R_l^{V,0}u_3,v_{l-1}-R_l^{V,0}v_{l-1})_{V^0}}{\|(v_{l-1},q_{l-1})\|_{X,l-1}}\\
&\quad\leq \|u_2-R_l^{V,0}u_3\|_{V^0}\sup_{v_{l-1}\in V_{l-1}}\frac{\|v_{l-1}-R_l^{V,0}v_{l-1}\|_{V^0}}{\|v_{l-1}\|_{V,l-1}}\preceq h\,\|u_2-R_l^{V,0}u_3\|_{V^0}.
\end{aligned}
$$

Next, we observe that

$$
\begin{aligned}
&B_l((\varphi_l,\psi_l)-R_l^{X,0}(\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1}),(0,I_{l-1}^{\widetilde{Q}}p_2)) \qquad (63)\\
&\quad= B_l((\varphi_l,\psi_l),(0,I_{l-1}^{\widetilde{Q}}p_2))-B_{l-1}((\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1}),(0,I_{l-1}^{\widetilde{Q}}p_2))=0.
\end{aligned}
$$

Using the variational problems we get

$$
\begin{aligned}
\|u_2-R_l^{V,0}u_3\|_{V^0}^2 &= B_l((\varphi_l,\psi_l),(u_2,p_2)-R_l^{X,0}(u_3,p_3))\\
&= B_l((\varphi_l,\psi_l),(u_2,p_2))-B_l((\varphi_l,\psi_l),R_l^{X,0}(u_3,p_3))\\
&= B_l((\varphi_l,\psi_l),(u_2,p_2))-B_{l-1}((\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1}),(u_3,p_3))\\
&= B_l((\varphi_l,\psi_l),(u_2,p_2))-B_l(R_l^{X,0}(\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1},(u_2,p_2))\\
&= B_l((\varphi_l,\psi_l)-R_l^{X,0}(\widetilde{\varphi}_l,\widetilde{\psi}_l),(u_2,p_2))\\
&= B_l((\varphi_l,\psi_l)-R_l^{X,0}(\widetilde{\varphi}_l,\widetilde{\psi}_l),(u_2,p_2-I_{l-1}^{\widetilde{Q}}p_2))\\
&\leq \|(\varphi_l,\psi_l)-R_l^{X,0}(\widetilde{\varphi}_l,\widetilde{\psi}_l)\|_{X,l}\ \ \|(u_2,p_2-I_{l-1}^{\widetilde{Q}}p_2)\|_{X,l}.
\end{aligned}
$$

With the consistency estimates onto the interpolation operators (13) and the prolongation operator (19), and the a priori estimate (15), we get

$$
\begin{aligned}
&\|(\varphi_l,\psi_l)-R_l^{X,0}(\widetilde{\varphi}_l,\widetilde{\psi}_l)\|_{X,l}\\
&\quad\preceq \|(\varphi_l,\psi_l)-R_l^{X,0}(\varphi_{l-1},\psi_{l-1})\|_{X,l}+\|R_l^{X,0}(\varphi_{l-1},\psi_{l-1})-R_l^{X,0}(\widetilde{\varphi}_{l-1},\widetilde{\psi}_{l-1})\|_{X,l}\\
&\quad\preceq h\,\|u_2-R_l^{V,0}u_3\|_{V^0}.
\end{aligned}
$$

Canceling one factor $\|u_2-R_l^{V,0}u_3\|_{V^0}$ gives the result.

□

Lemma 5
With the notation of (57) there holds

$$
\|(u_4,p_4)-R_l^{X,0}(u_3,p_3)\|_{X,l,\tilde{0}}\preceq\|(u_1,p_1)\|_{B_l}. \qquad (64)
$$

Proof: Norm equivalence (26) on $V_{l,T}$, orthogonality (21), stability (23) of $X_{l,T}$ and the definition of the projection (43) give

$$\begin{aligned}
&\|(u_4,p_4)-R_l^{X,0}(u_3,p_3)\|_{X,l,\tilde{0}} \simeq \|u_4-R_l^{V,0}u_3\|_{V,l}+\varepsilon^{1/2}\|p_4-p_3\|_{c_l}\\
&\preceq \|(u_4,p_4)-R_l^{X,0}(u_3,p_3)\|_{X,l} \preceq \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3-u_4,p_3-p_4),(v,q))}{\|(v,q)\|_{X,l}}\\
&= \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3,p_3),(v,q))}{\|(v,q)\|_{X,l}} = \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3,p_3-I_{l-1}^{\widetilde{Q}}p_3),(v,q))}{\|(v,q)\|_{X_l}}\\
&\preceq \|(R_l^{V,0}u_3,p_3-I_{l-1}^{\widetilde{Q}}p_3)\|_{X,l} \preceq \|(u_3,p_3-I_{l-1}^{\widetilde{Q}}p_3)\|_{X,l-1} \preceq \|(u_1,p_1)\|_{B_l},
\end{aligned}$$

and the proof if complete.

□

7 Application to nearly incompressible materials and Stokes

In [19] the case of nearly incompressible materials was investigated. We reformulate the results in the present framework. The problem is to find $u \in V := [H_0^1(\Omega)]^2$ such that

$$2\mu \int_\Omega e(u) : e(v)\,\mathrm{dx} + \lambda \int_\Omega \operatorname{div} u \operatorname{div} v\,\mathrm{dx} = \int_\Omega \tilde{f}^T v\,\mathrm{dx}, \tag{65}$$

with the positive constants λ and μ of Lamé, the strain operator $e(u) := 0.5(\nabla u + (\nabla u)^T)$ and the volume force $\tilde{f} \in [L_2(\Omega)]^2$. We are interested in the nearly incompressible case, i.e. the Poisson ration ν is close to 0.5. Then the 'bad parameter' $\varepsilon := 2\mu/\lambda$ becomes small. After dividing by 2μ, the primal bilinear-form $A(.,.)$, the operator Λ and the mixed bilinear-form are

$$\begin{aligned}
A(u,v) &= (e(u),e(v))_0 + \varepsilon^{-1}(\operatorname{div} u, \operatorname{div} v)_0,\\
\Lambda &= \operatorname{div},\\
B((u,p),(v,q)) &= (e(u),e(v))_0 + (\operatorname{div} u, q)_0 + (\operatorname{div} v, p)_0 - \varepsilon\,(p,q)_0.
\end{aligned}$$

The stronger and weaker spaces are $V^2 = H^2(\Omega)$, $Q^1 = H^1(\Omega)$ and $V^0 = L_2$. The regularity assumption (6) does hold on convex polygonal domains [11].
In [19] the stable pair of $P_2 - P_0$ finite element spaces was used, and the bilinear-forms $B_l(.,.)$ were set to $B(.,.)$. The norms $\|.\|_{V,l}$ and $\|.\|_{Q,l}$ the norms of H^1 and L_2.
The parameter h_l is the mesh size, the inverse inequality (11) and the interpolation operator (13) is standard. The a priori estimate (15) follows by the Aubin-Nitsche technique [12].
The primal bilinear-form on level l evaluates to

$$A_l(u,v) = (e(u),e(v))_0 + \varepsilon^{-1}(\overline{\operatorname{div} u}, \overline{\operatorname{div} v})_0,$$

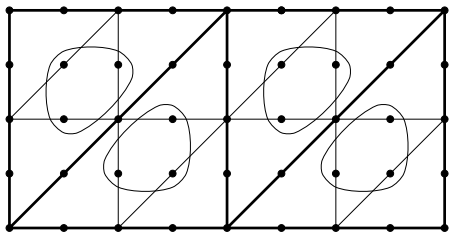


Figure 2: Subspace $V_{l,T}$ used in prolongation

where the $\bar{\cdot}$ does mean element-wise averaging.
The multigrid components are defined as follows. The trivial prolongation operator $R_l^{V,0}$ is the embedding operator. The spaces $V_{l,T}$ and $Q_{l,T}$ are defined as

$$V_{l,T} := \prod_{T \in \mathcal{T}_{l-1}} [H_0^1(T)]^2 \cap V_l \qquad \text{and} \qquad Q_{l,T} := \prod_{T \in \mathcal{T}_{l-1}} [L_2(T)/\mathbf{R}] \cap Q_l,$$

where $\mathcal{T}_l$ is the set of triangles at level l. The connected nodes of these spaces are drawn in Figure 2. The splitting (21) is done with $\widetilde{Q}_{l-1} = Q_{l-1}$.
The kernel of Λ_l has the base functions drawn in Figure 3. These leads to the definition of the n_l local spaces

$$V_{l,i} = [H_0^1(\Omega_i)]^2 \cap V_l \qquad \text{and} \qquad Q_{l,i} = [L_2(\Omega_i)/\mathbf{R}] \cap Q_l \tag{66}$$

assigned to the corner nodes $N_{l,i}$, of the triangulation. Ω_i is the union of elements in $\mathcal{T}_l$ adjacent to the node $N_{l,i}$. These space splitting is indicated in Figure 4.
The interpolation estimate (32) was proven in [19] with $\alpha = 1/2$.

Very popular for Stokes is the non-conforming $P_1 - P_0$ pairing [12]. It can be also used for nearly incompressible materials with pure Dirichlet boundary conditions [15].
The present analysis covers this element. The forms A_l and B_l are obtained by splitting the integrals over the triangles. The trivial prolongation $R_l^{V,0}$ is constructed by taking the mean value of the left and right values. The approximation (19) and property (25) are fulfilled. The spaces $X_{l,T}$ and $X_{l,i}$ for the prolongation and the smoother are defined analogue to the conforming case, and the analysis of the interpolation estimate (32) is similar to the conforming case.

8 Application to Reissner Mindlin plate

We use the Reissner Mindlin plate model with the discretization of [13]. In the primal space $V = H_0^1(\Omega) \times [H_0^1]^2$ we look for the transverse displacement w and the rotation vector β. The scaled problem is defined as

$$a(\beta, \eta) + \frac{k}{t^2}(\nabla w - \beta, \nabla v - \eta) = (g, v)_0 \qquad \forall\, (v, \eta) \in V$$

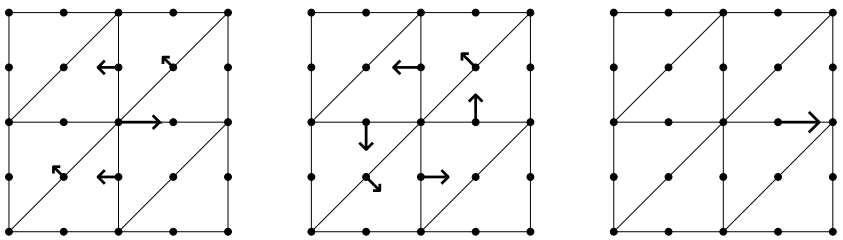


Figure 3: Basis functions for kernel of $I_l^Q \operatorname{div}$

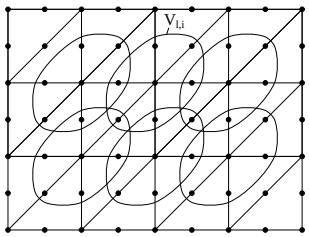


Figure 4: Subspaces containing div-free basis functions

with the bilinear form

$$a(\beta,\eta) = \frac{1}{6}\left[(e(\beta),e(\eta))_0 + \frac{\nu}{1-\nu}(\operatorname{div}\beta,\operatorname{div}\eta)_0\right].$$

The small parameter is the plate thickness t. The transverse load is Gt^3g. Further, k is the shear correction factor, ν the poisson ratio, G the shear modulus and $e(.)$ is the linear strain operator. We extend the linear functional $(g,.)_0$ onto V and set

$$\langle f,(v,\eta)\rangle = \langle g,v\rangle + \langle \delta,\eta\rangle \qquad \forall\,(v,\eta)\in V.$$

With the definitions

$$\begin{aligned}\Lambda &= \nabla w - \beta,\\ \varepsilon &= t^2/k\end{aligned}$$

the bilinear-form $A(.,.)$ has the structure of (2). The dual variable, the scaled shear force is

$$p = \varepsilon^{-1}\Lambda(w,\beta) = \frac{k}{t^2}(\nabla w - \beta).$$

The mixed form $B(.,.)$ evaluates to

$$B((w,\beta,p),(v,\eta,q)) = a(\beta,\eta) + (\nabla v - \eta,p)_0 + (\nabla w - \beta,q)_0 - \frac{t^2}{k}(p,q)_0$$

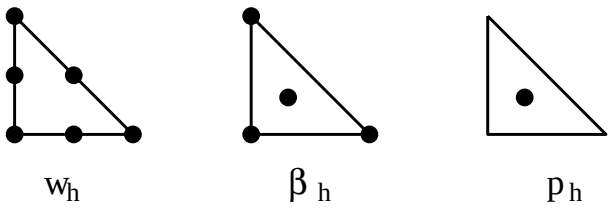


Figure 5: Finite elements for Reissner Mindlin

We define the norms

$$\begin{aligned}
\|(w,\beta)\|^2_{V^2} = \|w\|^2_{V^{2,w}} + \|\beta\|^2_{V^{2,\beta}} &= \inf_{w=w_0+w_r} \left\{ \|w_0\|^2_3 + t^{-2}\|w_r\|^2_2 \right\} + \|\beta\|^2_2, \\
\|(w,\beta)\|^2_{V^0} = \|w\|^2_{V^{0,w}} + \|\beta\|^2_{V^{0,\beta}} &= \inf_{w=w_0+w_r} \left\{ \|w_0\|^2_1 + t^{-2}\|w_r\|^2_0 \right\} + \|\beta\|^2_0, \\
\|p\|_{Q^1} &= \|p\|^2_0 + t^2\|p\|^2_1.
\end{aligned}$$

We evaluate the dual norm

$$\begin{aligned}
\|(g,\delta)\|_{(V^0)^*} &= \sup_{(v,\eta)\in V^0} \frac{\langle g, v\rangle + \langle \delta, \beta\rangle}{\left\{ \inf_{v=v_0+v_r} \|v_0\|^2_1 + t^{-2}\|v_r\|^2_0 + \|\beta\|^2_0 \right\}^{1/2}} \\
&= \sup_{v_0,v_r,\beta} \frac{\langle g, v_0\rangle + \langle g, v_r\rangle + \langle \delta, \eta\rangle}{\{\|v_0\|^2_1 + t^{-2}\|v_r\|^2_0 + \|\beta\|^2_0\}^{1/2}}
\end{aligned}$$

The supremum is taken at the solution (w_0, w_r, β) of the variational problem

$$(w_0, v_0)_1 + t^{-2}(w_r, v_r)_0 + (\beta, \eta)_0 = \langle g, v_0\rangle + \langle g, v_r\rangle + \langle \delta, \eta\rangle \qquad \forall\, v_0, v_r, \eta$$

and is equal to $\{\|w_0\|^2_1 + t^{-2}\|w_r\|^2_0 + \|\beta\|^2_0\}^{1/2} = \{\|g\|^2_{-1} + t^2\|g\|^2_0 + \|\delta\|^2_0\}^{1/2}$. Thus we computed

$$\|(g,\delta)\|^2_{(V^0)^*} = \|g\|^2_{-1} + t^2\|g\|^2_0 + \|\delta\|^2_0. \tag{67}$$

This is exactly the right hand side norm, for which [14] and [13] proved a solution in $V^2 \times Q^1$ and the regulaity estimate (6).
The discretization of [13] uses P_2 elements for the transverse displacement, P_1^+ for the rotations and P_0 for the shear, see Figure 5. The space P_1^+ are the piecewise linear functions enriched by the element bubbles $\lambda_1\lambda_2\lambda_3$ in barycentric coordinates. These spaces are non-nested due to the bubble functions. The form $B(.,.)$ is replaced by the mesh dependent bilinear-form

$$\begin{aligned}
B_l((w,\beta,p),(v,\eta,q)) &= B((w,\beta,p),(v,\eta,q)) \\
&+ \frac{\mu k}{t^2+h^2}\left(\nabla w - \beta - \frac{t^2}{k}p, \nabla v - \eta - \frac{t^2}{k}q\right)_0
\end{aligned}$$

with some constant $\mu \in (0,1)$. In [13] a local mesh size is used instead of the global one $h = h(l)$. The discrete shear operator can be computed locally and evaluates to

$$p_l = \varepsilon^{-1}\Lambda_l(w_l, \beta_l) = \frac{k}{t^2}(\overline{\nabla w_l - \beta_l}),$$

where $\bar{\cdot}$ is the element-wise averaging operator. The reduction to a positive definite problem gives the form

$$\begin{aligned} A_l((w_l,\beta_l),(v_l,\eta_l)) &= a(\beta_l,\eta_l) + \frac{k\mu}{h^2+t^2}(\nabla w_l - \beta_l, \nabla v_l - \eta_l)_0 \\ &+ \left(\frac{k}{t^2} - \frac{k\mu}{h^2+t^2}\right)(\overline{\nabla w_l - \beta_l}, \overline{\nabla v_l - \eta_l})_0 \end{aligned}$$

The discrete norms are

$$\begin{aligned} \|(w,\beta)\|_{V,l}^2 &= \|\beta\|_1^2 + \frac{1}{h^2+t^2}\|\nabla v - \beta\|_0^2, \\ \|p\|_{Q,l}^2 &= (h^2+t^2)\|p\|_0^2. \end{aligned}$$

We verify the approximation inequality (13) for the component-wise nodal interpolation operator $I_l^V = (I_l^{V,w}, I_l^{V,\beta})$. Let $w = w_0 + w_r$ with $\|w_0\|_3 + t^{-1}\|w_r\|_2 \preceq \|w\|_{V^{2,w}}$. Then

$$\begin{aligned} &\|w - I_l^{V,w}w\|_{V^{0,w}} + \|\beta - I_l^{V,\beta}\beta\|_{V^{0,\beta}} \\ &\quad\preceq \|w_0 - I_l^{V,w}w_0\|_1 + t^{-1}\|w_r - I_l^{V,w}w_r\|_0 + \|\beta - I_l^{V,\beta}\beta\|_0 \\ &\quad\preceq h^2\left\{\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\beta\|_2\right\} \preceq h^2\,\|(w,\beta)\|_{V^2}, \end{aligned}$$

and similar

$$\begin{aligned} &\|(w - I_l^{V,w}w, \beta - I_l^{V,\beta}\|_{V,l} \\ &\quad\preceq \|\beta - I_l^{V,\beta}\beta\|_1 + (h+t)^{-1}\|\nabla(w - I_l^{V,w}w) - (\beta - I_\beta^{V,\beta})\|_0 \\ &\quad\leq \|\beta - I_l^{V,\beta}\beta\|_1 + h^{-1}\|\beta - I_l^{V,\beta}\beta\|_0 + (h+t)^{-1}\left\{\|\nabla(w_0 - I_l^{V,w}w_0)\|_0 + \|\nabla(w_r - I_l^{V,w}w_r)\|_0\right\} \\ &\quad\leq h\,\|\beta\|_2 + \frac{h^2}{h+t}\|w_0\|_3 + \frac{ht}{h+t}t^{-1}\,\|w_r\|_2 \preceq h\,\|(w,\beta)\|_{V^2}. \end{aligned}$$

Define $P_{V,t}^w : V^w \to V_{l,t}^w$ as Clément interpolation operator into a finite element sub-space of H_0^1 of mesh size t. It fulfills the approximation estimate

$$\|w - P_{V,t}^w w\|_0 \preceq t\|w\|_1$$

and the continuity and inverse estimates

$$\|P_{V,t}^w w\|_k \preceq t^{m-k}\|w\|_m \qquad 0 \leq m \leq k \leq 1.$$

It provides an explicit splitting equivalent to the norm $\|w\|_{V^0}$ on V^w

$$\|P_{V,t}^w w\|_1 + t^{-1}\|w - P_{V,t}^w w\|_0 \simeq \|w\|_{V^{0,w}}. \tag{68}$$

The left hand side dominates the right by definition, the other direction is proved by taking a splitting $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^0}$ and

$$\begin{aligned}
&\|P^w_{l,t}w\|_1 + t^{-1}\|w - P^w_{V,t}w\|_0 \\
&\quad\preceq\quad \|P^w_{l,t}w_0\|_1 + \|P^w_{l,t}w_r\|_1 + t^{-1}\|w_0 - P^w_{V,t}w_0\|_0 + t^{-1}\|w_r - P^w_{V,t}w_r\|_0 \\
&\quad\preceq\quad \|w_0\|_1 + t^{-1}\|w_r\|_0 + t^{-1}t\,\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}.
\end{aligned}$$

It is used for the verification of the inverse estimate (11). First, we estimate

$$\begin{aligned}
\frac{1}{h+t}\|w_l\|_1 &\preceq \frac{1}{h+t}\|P^w_{l,t}w_l\|_1 + \frac{1}{h+t}\|(I - P^w_{l,t})w_l\|_1 \\
&\preceq \frac{1}{h}\|P^w_{l,t}w_l\|_1 + \frac{1}{h+t}(h^{-1} + t^{-1})\|(I - P^w_{l,t})w_l\|_0 \\
&= h^{-1}\left(\|P^w_{l,t}w_l\|_1 + t^{-1}\|(I - P^w_{l,t})w_l\|_0\right) \preceq h^{-1}\|w\|_{V^{0,w}}
\end{aligned}$$

using the inverse estimate $\|v\|_1 \preceq (h^1 + t^{-1})\|v\|_0$ on the space $V_l \cup V_{l,t}$. Then we get

$$\|(w_l, \beta_l)\|_{V,l} \preceq \|\beta_l\|_1 + (h+t)^{-1}\|\nabla w_l - \beta_l\|_0 \preceq h^{-1}\|\beta_l\|_0 + (h+t)^{-1}\|w_l\|_1 \preceq \|(w_l, \beta_l)\|_{V^0}.$$

Applying the duality trick in [13], Proposition 5, directly with the inner product from V_0 we obtain the a priori estimate (15).
Next, we start with the multigrid components. Let the trivial prolongation $R_l^{V,0}$ map the vertical displacement w_{l-1} to $w_l = w_{l-1}$, and map the rotations β_{l-1} to β_l by piecewise linear interpolation (and forget the bubbles). This prolongation is accurate enough for (19). The space splitting (21) is done by setting

$$Q_{l,T} = Q_l \qquad \text{and} \qquad \tilde{Q}_{l-1} = \{0\}.$$

Stability (23) is obtained by using all bubble functions for the rotations, i.e.

$$V_{l,T} = \{(0, \beta_l) \in V_l : \beta_l|_{\partial T} = 0 \ \ \forall\, T \in \mathcal{T}_l\}.$$

The stability proof of [13], Proposition 3, directly applies to the space $X_{l,T}$. The inclusion (24) and property (25) are trivially fulfilled. The norm equivalence (26) follows from Friedrichs' inequality plus scaling.
The space splitting for the smoother is done as

$$V_{l,i} = [H^1_0(\Omega_i)]^2 \cap V_l \qquad \text{and} \qquad Q_{l,i} = L_2(\Omega_i) \cap Q_l, \tag{69}$$

where each index i is associated with one corner node $N_{l,i}$. The set Ω_i is the union of elements sharing the node $N_{l,i}$. Again, the stability condition (28) is proved as in [13], the inclusion (29) is trivial.
The norm $\|.\|_{V,l,\tilde{0}}$ used in the multigrid analysis evaluates to

$$\|(w_l, \beta_l)\|^2_{V,l,\tilde{0}} := h^{-2} \inf_{w_l = w_0 + w_r} \left\{\|w_0\|_1^2 + t^{-2}\|w_r\|_0^2\right\} + h^{-2}\|\beta_l\|_0^2 + \frac{k}{t^2}\|\overline{\nabla w_l - \beta_l}\|_0^2, \tag{70}$$

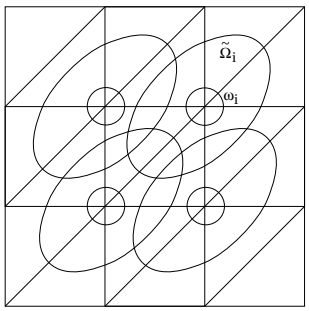
and the interpolation estimate (32) is the result of the following theorem:

Theorem 5
The splitting estimate

$$\|(w,\beta)\|_{[A,D]_{1/2}} \preceq \|(w,\beta)\|_{V,l,\tilde{0}} \tag{71}$$

is valid for all $(w,\beta) \in V_l$.

Proof: For the construction of the splitting we cover the domain with overlapping subdomains $\tilde{\Omega}_i$ contained in the sets Ω_i of triangles sharing the corner node N_i. The sets ω_i contain the nodes N_i, have measure $|\omega_i| \succeq |\Omega_i|$ and fulfill $\tilde{\Omega}_j \cap \omega_i = \emptyset$ for $i \neq j$, see the figure below.



Let $\{\Psi_i\}$ be a partition of unity fulfilling

$$\sum \Psi_i = 1, \qquad \text{supp}\, \Psi_i \subset \tilde{\Omega}_i,$$

$$h^2 \|\Psi_i\|_{2,\infty} + h\|\Psi_i\|_{1,\infty} + \|\Psi_i\|_{0,\infty} \preceq 1.$$

We follow [2] and construct the Fortin - operators

$$\begin{aligned} P^w_{V,l} &= P^{w,1}_{V,l} + P^{w,2}_{V,l}(id - P^{w,1}_{V,l}), \\ P^\beta_{V,l} &= P^{\beta,1}_{V,l} + P^{\beta,2}_{V,l}(id - P^{\beta,1}_{V,l}), \end{aligned}$$

where $P^{w,1}_{V,l}$ and $P^{\beta,1}_{V,l}$ are Scott-Zhang type projection operatos [20] based on the sets ω_i. The projector $P^{w,2}_{V,l}$ adjusts the edge bubbles such that

$$(P^{w,2}_{V,l} w)(N_i) = 0 \qquad \text{and} \qquad \int_{E_j} w - P^{w,2}_{V,l} w = 0$$

for corner nodes N_i and edges E_j. The projector $P^{\beta,2}_{V,l}$ adjusts the element bubbles such that

$$(P^{\beta,2}_{V,l} w)(N_i) = 0 \qquad \text{and} \qquad \int_{T_j} \beta - P^{\beta,2}_{V,l} \beta = 0$$

for corner nodes N_i and triangles T_j.
The constructed operators are projections, are continuous in the norms

$$\|P^w_{V,l}\|_1 \preceq 1, \qquad \|P^\beta_{V,l}\|_1 \preceq 1, \quad \|P^\beta_{V,l}\|_0 \preceq 1,$$

do approximate as good as

$$\begin{array}{rcll} \|w - P^w_{V,l} w\|_k & \preceq & h^{m-k} \|w\|_m & \forall\, w \in H^m_0, \qquad 0 \le k < m \le 2, \\ \|\beta - P^\beta_{V,l} \beta\|_0 & \preceq & h \|\beta\|_1 & \forall\, \beta \in H^1_0, \end{array} \tag{72}$$

and fulfill

$$\overline{\nabla w - \beta} = \overline{\nabla P^w_{V,l} w - P^\beta_{V,l} \beta},$$

as in [2].
Define $P_l^{H^2}$ as Clément interpolation operator into some finite element sub-subspace of H^2_0 of the same mesh-size h fulfilling the approximation

$$\|w - P_l^{H^2} w\|_k \preceq h^{m-k} \|w\|_m \quad \forall\, w \in H^m_0, \quad 0 \le k < m \le 2,$$

and the continuity and inverse estimate

$$\|P_l^{H^2} w\|_k \preceq h^{m-k} \|w\|_m \quad \forall\, w \in H^m_0, \quad 0 \le m \le k \le 2.$$

Then define the linear operator

$$T : H^2_0 \to V_h, \qquad T\tilde{w} = (P^w_{V,l} P_l^{H^2} \tilde{w}, P^\beta_{V,l} \nabla P_l^{H^2} \tilde{w}) \tag{73}$$

We use the triangle inequality and $A \preceq D$ to estimate

$$\begin{array}{rcl} \|(w,\beta)\|_{[A,D]_{1/2}} & \le & \|T P^w_{V,t} w\|_{[A,D]_{1/2}} + \|(w,\beta) - T P^w_{V,t} w\|_{[A,D]_{1/2}} \\ & \le & \|T P^w_{V,t} w\|_{[A,D]_{1/2}} + \|(w,\beta) - T P^w_{V,t} w\|_D. \end{array}$$

The prove of the theorem is complete after estimating the two terms in the following two Lemmata. □

Lemma 6
The estimate

$$\|T P^w_{V,t} w\|_{[A,D]_{1/2}} \preceq \|(w,\beta)\|_{V,l,\tilde{0}} \tag{74}$$

is valied.

Proof: First, we estimate

$$\|T\tilde{w}\|_D \preceq h^{-2} \|\tilde{w}\|_0 \qquad \forall\, \tilde{w} \in L_2 \tag{75}$$

Using the partition of unity we define

$$\varphi_i = \Psi_i P_l^{H^2} \tilde{w}$$

giving the splitting

$$P_l^{H^2} \tilde{w} = \sum_i \varphi_i,$$

what is L_2 stable

$$\sum_i \|\varphi_i\|_0^2 \preceq \|P_l^{H^2}\tilde{w}\|_0^2.$$

Due to the smoothness of Ψ_i the estimate

$$h^{-1}\|\varphi_i\|_1 + h^{-2}\|\varphi_i\|_2 \preceq \|\varphi_i\|_0$$

is valid. The domain decomposition and the interpolation operators have been design such that

$$(P_{V,l}^w\varphi_i, P_{V,l}^\beta\nabla\varphi_i) \in V_{l,i}.$$

So,

$$\begin{aligned}
\|T\tilde{w}\|_D^2 &\preceq \sum_i \|(P_{V,l}^w\varphi_i, P_{V,l}^\beta\nabla\varphi_i\|_A^2 \\
&\preceq \sum \|P_{V,l}^\beta\nabla\varphi_i\|_1^2 + h^{-2}\|\nabla P_{V,l}^w\varphi_i - P_{V,l}^\beta\nabla\varphi_i\|_0^2 + h^{-2}\|\overline{\nabla P_{V,l}^w\varphi_i - P_{V,l}^\beta\nabla\varphi_i}\|_0^2 \\
&\preceq \sum h^{-4}\|\varphi_i\|_0^2 + 0 \\
&\preceq h^{-4}\|\tilde{w}\|_0^2.
\end{aligned}$$

Then, we estimate

$$\|T\tilde{w}\|_A \preceq \|\tilde{w}\|_2 \qquad \forall\, \tilde{w} \in H_0^2 \tag{76}$$

by

$$\begin{aligned}
\|T\tilde{w}\|_A &\preceq \|P_{V,l}^\beta\nabla P_l^{H^2}\tilde{w}\|_1^2 + h^{-2}\|\nabla P_{V,l}^w P_l^{H^2}\tilde{w} - P_{V,l}^\beta\nabla P_l^{H^2}\tilde{w}\|_0 + 0 \\
&\preceq \|\tilde{w}\|_2^2 + h^{-2}\|\nabla(P_{V,l}^w - id)P_l^{H^2}\tilde{w}\|_0^2 + h^{-2}\|(P_{V,l}^\beta - id)\nabla P_l^{H^2}\tilde{w}\|_0^2 \\
&\preceq \|\tilde{w}\|_2^2.
\end{aligned}$$

Applying operator interpolation to (75) and (76) we conclude

$$\|T\tilde{w}\|_{[A,D]_{1/2}} \preceq h^{-1}\|\tilde{w}\|_1 \qquad \forall\, \tilde{w} \in H_0^1. \tag{77}$$

The necessary interpolation estimate $\|.\|_{H_0^1} \simeq \|.\|_{[H_0^2,L_2]_{1/2}}$ is found in [6].
We split $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}$ and finish the lemma by

$$\begin{aligned}
\|TP_{V,t}^w w\|_{[A,D]_{1/2}} &\preceq h^{-1}\|P_{V,t}^w w\|_1 \\
&\preceq h^{-1}\|P_{V,t}^w w_0\|_1 + h^{-1}\|P_{V,t}^w w_r\|_1 \\
&\preceq h^{-1}\|w_0\|_1 + h^{-1}t^{-1}\|w_r\|_0 \\
&\preceq h^{-1}\|w\|_{V^{0,w}} \preceq \|(w,\beta)\|_{V,l,\tilde{0}}.
\end{aligned}$$

Lemma 7
The estimate

$$\|TP_{V,t}^w w - (w,\beta)\|_D \preceq \|(w,\beta)\|_{V,l,\tilde{0}} \tag{78}$$

is valid.

First, we prove for all $(v,\eta) \in V_l$ the splitting estimate

$$\|(v,\eta)\|_D \preceq h^{-1}\|\eta\|_0 + h^{-1}(h+t)^{-1}\|v\|_0 + t^{-1}\|\overline{\nabla v - \eta}\|_0. \tag{79}$$

The case $h \preceq t$ is trivial, and we may assume $t \preceq h$. We set

$$\begin{aligned} v_i &= P^w_{V,l}(\Psi_i v), \\ \eta_i &= P^\beta_{V,l}(\Psi_i \eta) + P^{\beta,2}_{V,l}\left[\nabla(\Psi_i v) - (\Psi_i \eta)\right], \end{aligned}$$

such that we can represent

$$\begin{aligned} v &= \sum v_i, \\ \eta &= \sum \eta_i + P^{\beta,2}_{V,l}(\eta - \nabla v). \end{aligned}$$

Because $P^{\beta,2}_{V,l}(\eta - \nabla v)$ contains only A-orthogonal bubble functions, which can be split locally, we get

$$\|(0, P^{\beta,2}_{V,l}(\eta - \nabla v)\|_D \preceq \|(0, P^{\beta,2}_{V,l}(\eta - \nabla v))\|_A \preceq h^{-2}\|v\|_0 + h^{-1}\|\eta\|_0 + t^{-1}\|\overline{\nabla v - \eta}\|_0.$$

We estimate

$$\begin{aligned} \|(v_i,\eta_i)\|^2_A &\preceq \|\eta_i\|_1 + h^{-2}\|\nabla v_i - \eta_i\|^2_0 + \|\overline{\nabla v_i - \eta_i}\|_0 \\ &\preceq h^{-2}\|(\Psi_i v)\|_0 + h^{-1}\|(\Psi_i \eta)\|_0 + 0. \end{aligned}$$

Thus, we obtain estimate (79) by

$$\begin{aligned} \|(v,\eta)\|^2_D &\preceq \sum_i \|(v_i,\eta_i)\|^2_A + \|(0, P^{\beta,2}_{V,l}(\eta - \nabla v)\|^2_D \\ &\preceq \sum_i h^{-4}\|(\Psi_i v)\|^2_0 + h^{-2}\|(\Psi_i \eta)\|^2_0 + \|(0, P^{\beta,2}_{V,l}(\eta - \nabla v)\|^2_D \\ &\preceq h^{-4}\|v\|^2_0 + h^{-2}\|\eta\|^2_0 + t^{-2}\|\overline{\nabla v - \eta}\|^2_0. \end{aligned}$$

We split $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}$ and estimate

$$\begin{aligned} \|(TP^w_{V,t}w)_w - w\|_0 &\preceq \|P^w_{V,l}P^{H^2}_l P^w_{V,t}w_0 - w_0\|_0 + \|P^w_{V,l}P^{H^2}_l P^w_{V,t}w_r - w_r\|_0 \\ &\preceq (h+t)\|w_0\|_1 + t\,t^{-1}\|w_r\|_0 \preceq (h+t)\|w\|_{V^{0,w}}, \end{aligned}$$

and, using (68),

$$\|(TP^w_{V,t})_\beta\|_0 \preceq \|P^\beta_{V,l}\nabla P^{H^2}_l P^w_{V,t}w\|_0 \simeq \|P^w_{V,t}w\|_1 \preceq \|w\|_{V^{0,w}}.$$

Combining (79) with the last two estimates, and using $\overline{\nabla(T)_w - (T)_\beta} = 0$, we obtain estimate (78), and the proof is complete $\square$

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