SOLUTION OF ONE-TIME-STEP PROBLEMS IN ELASTOPLASTICITY BY A SLANT NEWTON METHOD*

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Abstract. We discuss a solution algorithm for quasi-static elastoplastic problems with hardening. Such problems can be described by a time dependent variational inequality, where the displacement and the plastic strain fields serve as primal variables. After discretization in time, one variational inequality of the second kind is obtained per time step and can be reformulated as each one minimization problem with a convex energy functional which depends smoothly on the displacement and non-smoothly on the plastic strain. There exists an explicit formula how to minimize the energy functional with respect to the plastic strain for a given displacement. By substitution, the energy functional can be written as a functional depending only on the displacement. The theorem of Moreau from convex analysis states that the energy functional is differentiable with an explicitly computable first derivative. The second derivative of the energy functional does not exist, hence the plastic strain minimizer is not differentiable on the elastoplastic interface, which separates the continuum in elastically and plastically deformed parts. A Newton-like method exploiting slanting functions of the energy functional's first derivative instead of the nonexistent second derivative is applied. Such method is called a slant Newton method for short. The local super-linear convergence of the slant Newton method in the discrete case is shown and sufficient regularity assumptions are formulated, which would guarantee the local super-linear convergence also in the continuous case.

Key words. plasticity, semismooth methods, slanting functions, Newton's method, Moreau's theorem

AMS subject classifications. 74C05, 74G15, 65J15, 65M60, 65H10, 49M15

1. Introduction. We consider a quasi-static initial-boundary value problem for small strain elastoplasticity with hardening. Throughout the paper, only the linear isotropic hardening law is considered, however an extension to other kinds of linear hardening is straightforward. Several interesting computation techniques for solving the elastoplastic problem with various kinds of hardening can be found in [24, 4, 6, 27, 2, 23, 22]. For the efficient solution of problems without hardening we refer to [31, 32]. By adding the equilibrium of forces and the plastic flow law and subsequent integration over the body domain, we obtain a time dependent variational inequality. Existence and uniqueness results concerning the solution of such inequality have been proved with respect to different models of hardening in [19, 18, 20, 21, 5, 15, 16]. Therefore, results concerning general variation inequalities [8] have been used.

The traditional numerical methods for solving the time dependent variational inequality are based on the explicit Euler time-discretization with respect to the loading history. In this case the idea of implicit return mapping discretization [27] turned out fruitful for calculations. By an implicit Euler time-discretization, on the other hand, the time dependent inequality is approximated by a sequence of time independent variational inequalities of the second kind for the unknown displacement u and plastic strain p. Each of these inequalities is equivalent [11] to a minimization problem with a convex but non-smooth energy functional, $\bar{J}(u,p) \to \min$. It has been already shown in [6] that a method of alternating minimization convergences globally and linearly. The minimization with respect to the plastic strain can be calculated locally by using an explicitly known dependence [2] of the plastic strain on the total strain, i.e.,

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$p = \tilde{p}(\varepsilon(u))$. Thus, the equivalent energy minimization problem for the displacement u only,

$$J(u) := \bar{J}(u, \tilde{p}(\varepsilon(u)) \to \min\,,$$

can be defined. Since the dependencies of the energy functional on the second argument, and of the minimizer $\tilde{p}$ on the total strain $\varepsilon(u)$ are not smooth, the Fréchet derivative $D\,J(u)$ seems not to exist. However, a multiplicative additive Schwarz method [6] and a (damped) quasi Newton scheme [2] are shown to converge globally and linearly. The super-linear convergence is discussed but not proved in the latter article.

The main theoretical contribution of this paper is the extension of the analysis done in [6, 2]. We show that the structure of the energy functional $J(u)$ satisfies the assumptions of Moreau's theorem from convex analysis and therefore, the energy functional $J(u)$ is Fréchet differentiable (Corollary 3.6 on page 6) with the explicitly computable Fréchet derivative $D\,J(u)$. However, the second derivative of the energy functional, $D^{\,2}J(u)$, does not exist because of the non-differentiability of the plastic strain minimizer $\tilde{p}$ on the elastoplastic interface, which separates the deformed continuum in elastically and plastically deformed parts.

By the concept of slant differentiability, introduced by X. Chen, Z. Nashed and L. Qi in [7], we define a Newton-like method using slanting functions instead of the usual derivative. We call such method a slant Newton method for short. One of the main results in [7] is, that a slant Newton method converges locally super-linear under the same assumptions as the classical Newton method. The main task is to find slanting functions for the mapping $\max\{0,\cdot\}$, which occurs within the formula of the plastic minimizer $\tilde{p}$ and causes its non-differentiability. Such slanting functions are easy to find in the spacial discrete case, e. g. after the FEM discretization. Theorem 4.14 on page 16 provides an explanation to an open question formulated in [2, Remark 7.5] concerning the super-linear convergence.

The spatially continuous case is more complicated and requires some extra regularity assumptions for the trial stress in each slant Newton step. To the best knowledge of the authors, there are no theoretical results yet known, which would guarantee the required regularity properties. Already existing regularity results, e. g. such as in [10, 3], concern the regularity of the stress and displacement fields which solve the elastoplastic one-time-step problem, but not of the trial stresses during a slant Newton iteration. Let us mention that iteration techniques were successfully used to prove regularity results for some smoothed initial boundary value problems of the plastic flow theory in [25] (see also [24]). Thus, our work may serve as a starting point for more regularity analysis concerning elastoplastic problems.

A numerical experiment concludes the paper. For the space-discretization, the finite element method of the lowest order with piece-wise linear nodal basis-functions for the FE-solution u_h are used. This experiment, as well as all of the others done by the authors, e. g. in [14], provide the following conclusions:

(i) The slant Newton method converges super-linearly at all levels of refinement.
(ii) The number of iteration steps does hardly depend on the level of refinement.

2. Mathematical Modeling. Let $\Theta := [0, T]$ be a time interval, and let Ω be a bounded domain in the space $\mathbb{R}^3$ with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. The equilibrium of forces in the quasi-static case reads

$$-\mathrm{div}\,(\sigma(x,t)) = f(x,t) \qquad \forall\ (x,t) \in \Omega \times \Theta\,, \tag{2.1}$$

where $\sigma(x,t) \in \mathbb{R}^{3\times3}$ is called Cauchy's stress tensor and $f(x,t) \in \mathbb{R}^3$ represents the volume force acting at the material point $x \in \Omega$ at the time $t \in \Theta$. Let $u(x,t) \in \mathbb{R}^3$ denote the displacement of the body, and let

$$\varepsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \tag{2.2}$$

be the linearized Green-St. Venant strain tensor. In elastoplasticity, the total strain ε is split additively into an elastic part e and a plastic part p, that is,

$$\varepsilon = e + p\,. \tag{2.3}$$

We assume a linear dependence of the stress on the elastic part of the strain, which is defined by Hooke's law

$$\sigma = \mathbb{C}\, e, \tag{2.4}$$

where the single components of the elastic stiffness tensor $\mathbb{C} \in \mathbb{R}^{3\times3\times3\times3}$ are defined $\mathbb{C}_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Here, $\lambda > 0$ and $\mu > 0$ denote the Lamé constants, and δ_{ij} the Kronecker-symbol. Instead of Lamé constants, one sometimes prescribes Young's modulus $E = \mu\,(3\lambda + 2\mu)\,/\,(\lambda + \mu)$ and Poisson's ratio $\nu = \lambda/\,(2\lambda + 2\mu)$.

Let the boundary Γ be split into a Dirichlet part Γ_D and a Neumann part Γ_N, which satisfy $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$. We assume the boundary conditions

$$u = u_D \ \text{ on } \Gamma_D \quad \text{and} \quad \sigma\, n = g \ \text{ on } \Gamma_N\,, \tag{2.5}$$

where $n(x,t)$ denotes the exterior unit normal, $u_D(x,t) \in \mathbb{R}^3$ denotes a prescribed displacement and $g(x,t) \in \mathbb{R}^3$ denotes a prescribed traction force. If $p = 0$ in (2.3), the system (2.1) – (2.5) describes the elastic behavior of the continuum Ω.

Two more properties, incorporating the admissibility of the stress σ with respect to a certain hardening law and the time evolution of the plastic strain p, are required. Therefore, we introduce the hardening parameter α and define the generalized stress (σ, α), which we call admissible if for a given convex yield functional ϕ there holds

$$\phi(\sigma, \alpha) \le 0\,. \tag{2.6}$$

The explicit form of ϕ depends on the choice of a certain hardening law (see Remark 2.2). The second, specifically elastoplastic, property addresses the time development of the generalized plastic strain $(p, -\alpha)$. There must hold the normality condition

$$\langle(\dot{p}, -\dot{\alpha})\,,\,(\tau, \beta) - (\sigma, \alpha)\rangle_F \le 0 \qquad \forall\,(\tau, \beta) \text{ which satisfy } \phi(\tau, \beta) \le 0\,, \tag{2.7}$$

where $\dot{p}$ and $\dot{\alpha}$ denote the first time derivatives of p and α. Therefore, we need initial conditions, which read

$$p(x,0) = p_0(x) \quad \text{and} \quad \alpha(x,0) = \alpha_0(x) \qquad \forall\, x \in \Omega\,, \tag{2.8}$$

with given initial values $p_0 : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}}$ and $\alpha_0 : \Omega \to [0, \infty[$.

PROBLEM 2.1 (classical formulation). *Find* (u, p, α), *which satisfies* (2.1) – (2.8).

REMARK 2.2. In this paper we concentrate on the isotropic hardening law, where the hardening parameter α is a scalar function $\alpha : \Omega \to \mathbb{R}$ and the yield functional ϕ is defined by

$$\phi(\sigma, \alpha) := \begin{cases} \|\operatorname{dev}\sigma\|_F - \sigma_y(1 + H\alpha) & \text{if } \alpha \ge 0, \\ +\infty & \text{if } \alpha < 0. \end{cases} \tag{2.9}$$

Here, the Frobenius norm $\|A\|_F := \langle A\,,\, A\rangle_F^{1/2}$ is defined by the matrix scalar product $\langle A\,,\, B\rangle_F := \sum_{ij} a_{ij} b_{ij}$ for $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$ and $B = (b_{ij}) \in \mathbb{R}^{3\times 3}$. The deviator is defined for square matrices by $\operatorname{dev} A = A - \frac{\operatorname{tr} A}{\operatorname{tr} I} I$, where the trace of a matrix is defined by $\operatorname{tr} A = \langle A\,,\, I\rangle_F$ and I denotes the identity matrix. The real material constants $\sigma_y > 0$ and $H > 0$ are called yield stress and modulus of hardening, respectively.

We turn to the specification of proper function spaces. For a fixed time $t \in \Theta$, let

$$u \in V := \left[H^1(\Omega)\right]^3\,, \quad p \in Q := [L_2(\Omega)]^{3\times 3}_{\text{sym}}\,, \quad \alpha \in L_2(\Omega)\,.$$

We define the hyper plains $V_D := \{v \in V \mid v_{|\Gamma_D} = u_D\}$ and $V_0 := \{v \in V \mid v_{|\Gamma_D} = 0\}$, and the associated scalar products and norms

$$\langle u\,,\, v\rangle_V := \int_\Omega \left(u^T v + \langle \nabla u\,,\, \nabla v\rangle_F\right)\, \mathrm{d}x\,, \qquad \|v\|_V := \langle v\,,\, v\rangle_V^{1/2}\,,$$

$$\langle p\,,\, q\rangle_Q := \int_\Omega \langle p\,,\, q\rangle_F\, \mathrm{d}x\,, \qquad \|q\|_Q := \langle q\,,\, q\rangle_Q^{1/2}\,.$$

Starting from Problem 2.1, one can derive a uniquely solvable time dependent variational inequality for unknown displacement $u \in H^1(\Theta; V_D)$ and plastic strain $p \in H^1(\Theta; Q)$ (see [16, Theorem 7.3] for details). However, the numerical treatment requires a time discretization of this variational inequality. Therefore, we pick a fixed number of time tics $0 = t_0 < t_1 < \ldots < t_{N_\Theta} = T$ out of Θ. We introduce the notation

$$u_k := u(t_k)\,,\;\; p_k := p(t_k)\,,\;\; \alpha_k := \alpha(t_k)\,,\;\; f_k := f(t_k)\,,\;\; g_k := g(t_k)\,,\;\; \ldots\,,$$

and approximate time derivatives by the backward difference quotients

$$\dot{p}_k \approx (p_k - p_{k-1}) \,/\, (t_k - t_{k-1}) \quad \text{and} \quad \dot{\alpha}_k \approx (\alpha_k - \alpha_{k-1}) \,/\, (t_k - t_{k-1})\,.$$

Consequently, the time dependent problem is approximated by a sequence of time independent variational inequalities of the second kind. Each of these variational inequalities can be equivalently expressed by a minimization problem, which by definition of the set of extended real numbers, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, reads in the case of isotropic hardening [6, Example 4.5]:

PROBLEM 2.3 (One-time-step problem). *Let $k \in \{1, \ldots, N_\Theta\}$ denote a given time step, let $p_{k-1} \in Q$ and $\alpha_{k-1} \in L_2(\Omega)$ be fixed arbitrarily such, that $\alpha_{k-1}(x) \geq 0$ holds almost everywhere. Define $\bar{J}_k : V \times Q \to \overline{\mathbb{R}}$ by $\bar{J}_k(v, q) := +\infty$ if $\operatorname{tr} q \neq \operatorname{tr} p_{k-1}$, else*

$$\bar{J}_k(v, q) := \frac{1}{2} \int_\Omega \langle \mathbb{C}(\varepsilon(v) - q)\,,\, \varepsilon(v) - q\rangle_F + (\alpha_{k-1} + \sigma_y H \|q - p_{k-1}\|_F)^2\, \mathrm{d}x + \int_\Omega \sigma_y \|q - p_{k-1}\|_F\, \mathrm{d}x - \int_\Omega f_k^T\, v\, \mathrm{d}x - \int_{\Gamma_N} g_k^T\, v\, \mathrm{d}s\,. \tag{2.10}$$

Find $(u_k, p_k) \in V_D \times Q$ such, that $\bar{J}_k(u_k, p_k) \leq \bar{J}_k(v, q)$ holds for all $(v, q) \in V_D \times Q$.

The convex functional $\bar{J}_k$ expresses the mechanical energy of the deformed system at the kth time step. Notice, that $\bar{J}_k$ is smooth with respect to the test displacements v, but not with respect to the test plastic strains q. Problem 2.3 has a unique solution (see, e. g., [9, Proposition 1.2 in Chapter II]).

The hardening parameter $\alpha_k \in L_2(\Omega)$ does not appear in Problem 2.3 directly, but can be calculated analytically in dependence on the plastic strain by $\alpha_k = \tilde{\alpha}_k(p_k)$, where $\tilde{\alpha}_k : Q \to L_2(\Omega)$ reads [6, Example 4.5]

$$\tilde{\alpha}_k(q) = \alpha_{k-1} + \sigma_y H \|q - p_{k-1}\|_F\,, \tag{2.11}$$

in the case of isotropic hardening.

3. Derivation of a Smooth Minimization Problem in the Displacement. Various strategies have been introduced to solve the minimization in Problem 2.3. C. Carstensen [6] proved the global linear convergence of one solution algorithm for Problem 2.3 in the spatially discrete case. A separated minimization in v and q is presented, where q is minimized by an explicit formula [2] (cf. Theorem 3.8 on page 7) and a domain decomposition method [6] or a quasi Newton method [2] are applied to solution in v.

Similar to [2, § 7], we study a minimization problem with respect to v only, which one derives from Problem 2.3 by the substitution of the known minimizer for q in the energy functional. An important observation will be, that the resulting functional is smooth and its derivative is explicitly computable. To discuss this issue, we introduce a more abstract formulation of (2.10). Therefore, we define the $\mathbb{C}$-scalar-product, the $\mathbb{C}$-norm, a convex functional ψ_k and a linear functional l_k by

$$\langle q_1 \,,\, q_2\rangle_{\mathbb{C}} := \int_\Omega \langle \mathbb{C}\, q_1(x) \,,\, q_2(x)\rangle_F \;\mathrm{d}x\,, \qquad \|q\|_{\mathbb{C}} \;:=\; \langle q \,,\, q\rangle_{\mathbb{C}}^{1/2}\,, \tag{3.1}$$

$$\psi_k(q) := \begin{cases} \int_\Omega \left(\frac{1}{2}\tilde{\alpha}_k(q)^2 + \sigma_y\|q - p_{k-1}\|_F\right)\,\mathrm{d}x & \text{if } \operatorname{tr} q = \operatorname{tr} p_{k-1}\,, \\ +\infty & \text{else}\,, \end{cases} \tag{3.2}$$

$$l_k(v) := \int_\Omega f_k^T\, v\;\mathrm{d}x + \int_{\Gamma_N} g_k^T\, v\;\mathrm{d}s\,, \tag{3.3}$$

where $\tilde{\alpha}_k(q)$ is defined in (2.11). Then the functional $\bar{J}_k(v,q)$ in (2.10) rewrites:

$$\bar{J}_k(v,q) = \frac{1}{2}\|\varepsilon(v) - q\|_{\mathbb{C}}^2 + \psi_k(q) - l_k(v)\,. \tag{3.4}$$

The following results are formulated for functions mapping from a Hilbert space $\mathcal{H}$ into the set of extended real numbers $\overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$. The Hilbert space $\mathcal{H}$ provides a scalar product $\langle \circ \,,\, \diamond\rangle_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}} := \langle \cdot \,,\, \cdot\rangle_{\mathcal{H}}^{1/2}$. The topological dual space of $\mathcal{H}$ is denoted by $\mathcal{H}^*$. Further, if a function F is Fréchet differentiable, we will denote its derivative in a point x by $D\,F(x)$ and its Gâteaux differential in the direction y by $D\,F(x\,;\,y)$. A couple of definitions are required to formulate the next results.

Definition 3.1. A mapping $F : \mathcal{H} \to \overline{\mathbb{R}}$ is said to be *convex* if, for every x and y in $\mathcal{H}$, we have

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y) \qquad \forall\, t \in [0,1]\;, \tag{3.5}$$

whenever the right hand side is defined.

Definition 3.2. A mapping $F : \mathcal{H} \to \mathbb{R}$ is said to be *strictly convex* if it is convex and the strict inequality holds in (3.5) for all $x, y \in \mathcal{H}$ with $x \neq y$ and for all $t \in \,]0,1[$.

Definition 3.3. A mapping $F : \mathcal{H} \to \overline{\mathbb{R}}$ is said to be *proper*, if

(i) there exists $x \in \mathcal{H}$ such, that $F(x) < +\infty$,

(ii) for all $x \in \mathcal{H}$ there holds $F(x) > -\infty$.

Definition 3.4. A mapping $F : \mathcal{H} \to \overline{\mathbb{R}}$ is said to be *lower semi continuous* at $x \in \mathcal{H}$ if

$$\lim_{y\to x} F(y) \geq F(x)\,.$$

F is said to be *lower semi continuous in* $\mathcal{H}$ if F is lower semi continuous at all $x \in \mathcal{H}$.

THEOREM 3.5 (Moreau). *Let the function* $f : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}$ *be defined*

$$f(x,y) = \frac{1}{2}\|x-y\|_{\mathcal{H}}^2 + \psi(x) \tag{3.6}$$

where ψ *is a convex, proper and lower semi continuous mapping of* $\mathcal{H}$ *into* $\overline{\mathbb{R}}$.

Then $F(y) := \inf_{x\in\mathcal{H}} f(x,y)$ *is well defined as a functional from* $\mathcal{H}$ *into* $\mathbb{R}$ *and there exists a unique mapping* $\tilde{x} : \mathcal{H} \to \mathcal{H}$ *such, that* $F(y) = f(\tilde{x}(y), y)$ *holds for all* $y \in \mathcal{H}$. *Moreover,* F *is strictly convex and Fréchet differentiable with the derivative*

$$D\,F(y) = \langle y - \tilde{x}(y)\,,\,\cdot\rangle_{\mathcal{H}} \in \mathcal{H}^* \qquad \forall\, y \in \mathcal{H}\,. \tag{3.7}$$

Proof. [26, Proposition 3.a, Proposition 7.d] ☐

The result, regarding the smoothness of F, is surprising, particularly since both mappings ψ and $\tilde{x}$ are in general not smooth. We apply Theorem 3.5 to the elastoplastic energy functional $\bar{J}_k$ in (3.4).

COROLLARY 3.6. *Let* $k \in \{1,\ldots,N_\Theta\}$ *denote the time step, and let* $\bar{J}_k$ *be defined as in* (3.1) – (3.4). *Then there exists a unique mapping* $\tilde{p}_k : Q \to Q$ *satisfying*

$$\bar{J}_k\,(v, \tilde{p}_k\,(\varepsilon\,(v))) = \inf_{q\in Q} \bar{J}_k\,(v,q) \quad \forall\, v \in V_D\,.$$

Let J_k *be a mapping of* V_D *into* $\mathbb{R}$ *defined as*

$$J_k(v) := \bar{J}_k(v, \tilde{p}_k(\varepsilon(v))) \quad \forall\, v \in V_D\,. \tag{3.8}$$

Then, J_k *is strictly convex and Fréchet differentiable. The associated Gâteaux differential* $D\,J_k : (V_D, V_0) \to \mathbb{R}$ *at* $v \in V_D$ *into the direction* $w \in V_0$ *reads*

$$D\,J_k(v\,;\,w) = \langle \varepsilon(v) - \tilde{p}_k(\varepsilon(v))\,,\,\varepsilon(w)\rangle_{\mathbb{C}} - l_k(w) \tag{3.9}$$

with the scalar product $\langle \circ\,,\,\diamond\rangle_{\mathbb{C}}$ *defined in* (3.1) *and* l_k *defined in* (3.3).

Proof. Using (3.1) – (3.4), we may rewrite functional $\bar{J}_k : V \times Q \to \overline{\mathbb{R}}$ by

$$\bar{J}_k(v,q) = f_k(\varepsilon(v), q) - l_k(v)\,,$$

where $f_k : Q \times Q \to \overline{\mathbb{R}}$ is defined

$$f_k(\varepsilon, q) := \frac{1}{2}\|\varepsilon - q\|_{\mathbb{C}}^2 + \psi_k(q)\,.$$

Theorem 3.5 states an existence of a unique minimizer $\tilde{p}_k : Q \to Q$ which satisfies

$$f_k(\varepsilon(v), \tilde{p}_k(\varepsilon(v))) = \inf_{q\in Q} f_k(\varepsilon(v), q)\,.$$

Moreover, it states that the functional $F_k(\varepsilon(v)) := f_k(\varepsilon(v), \tilde{p}_k(\varepsilon(v)))$ is strictly convex and differentiable with respect to $\varepsilon(v) \in Q$. Since $\varepsilon(\cdot)$ is a linear and injective mapping of V_D into Q, the compound functional $F_k(\varepsilon(v))$ is Fréchet differentiable and strictly convex with respect to $v \in V_D$. Considering the linearity of l_k, we conclude the strict convexity and Fréchet differentiability of J_k defined in (3.8). The explicit formula of the Gâteaux differential $D\,J_k(v\,;\,w)$ in (3.9) results from the linearity of the two mappings l_k and ε, and the Fréchet derivative $D\,F_k(\varepsilon(v)\,;\,\cdot) = \langle \varepsilon(v) - \tilde{p}_k(\varepsilon(v))\,,\,\cdot\rangle_{\mathbb{C}}$ as in (3.7). ☐

Corollary 3.6 already assures the unique existence of a plastic strain minimizer $\tilde{p}_k$. We now turn to the calculation of its explicit form, and define $\hat{p}_k(\cdot) := \tilde{p}_k(\cdot) - p_{k-1}$. For arbitrarily chosen $v \in V_D$, it is equivalent to find the minimizer $\tilde{p}_k(\varepsilon(v))$ of functional $\bar{J}_k(v, q)$ in (2.10) with respect to q, or to find the minimizer $\hat{p}_k(\varepsilon(v))$ of the functional

$$\frac{1}{2}\left(2\mu + \sigma_y^2 H^2\right) \|q\|_Q^2 - \langle \mathbb{C}\left(\varepsilon(v) - p_{k-1}\right), q\rangle_Q + \langle \sigma_y \left(1 + \alpha_{k-1} H\right), \|q\|_F\rangle_{L_2(\Omega)} \tag{3.10}$$

amongst trace-free elements $q \in Q$. The explicit form of $\hat{p}_k : Q \to Q$ is presented in the next theorem, which is almost identical to [2, Proposition 7.1], with the only difference, that here we investigate the plastic strain minimizer as a mapping from Q into Q instead of the pointwise version $\hat{P}_k : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}}$. But first of all, another definition is required.

DEFINITION 3.7. Let F be a mapping of $\mathcal{H}$ into $\overline{\mathbb{R}}$. F is said to be *subdifferentiable* at $x \in \mathcal{H}$ if there exists $x^* \in \mathcal{H}^*$ such that there holds

$$F(x+y) \geq F(x) + \langle x^*, y\rangle_{\mathcal{H}} \qquad \forall\, y \in \mathcal{H}.$$

We call x^* a *subgradient*, and the set of all subgradients in x is said to be the *subdifferential* of F in x, denoted by $\partial F(x)$.

For shorter writing, we skip the comment 'almost everywhere' in the following theorem, whenever mappings defined on Lebesgue spaces are evaluated pointwise.

THEOREM 3.8. *Let $A \in Q$, $b \in L_2(\Omega)$ with $b(x) > 0$ in Ω, and $\xi \in \mathbb{R}$ with $\xi > 0$. Then there exists exactly one $\hat{p} \in Q$ with $\|\operatorname{tr}\hat{p}\|_{L_2(\Omega)} = 0$, which satisfies*

$$\langle A - \xi\,\hat{p}, q - \hat{p}\rangle_Q \leq \langle b, \|q\|_F - \|\hat{p}\|_F\rangle_{L_2(\Omega)} \tag{3.11}$$

for all $q \in Q$ with $\|\operatorname{tr} q\|_{L_2(\Omega)} = 0$. This $\hat{p}$ is characterized as the minimizer of

$$\frac{\xi}{2}\|q\|_Q^2 - \langle A, q\rangle_Q + \langle b, \|q\|_F\rangle_{L_2(\Omega)} \tag{3.12}$$

amongst trace-free elements $q \in Q$, and reads

$$\hat{p} = \frac{1}{\xi}\max\{0, \|\operatorname{dev} A\|_F - b\}\,\frac{\operatorname{dev} A}{\|\operatorname{dev} A\|_F}. \tag{3.13}$$

The minimal value of (3.12)*, attained for $\hat{p}$ as in* (3.13)*, is*

$$-\frac{1}{2\,\xi}\,\|\max\{0, \|\operatorname{dev} A\|_F - b\}\|^2_{L_2(\Omega)}\,. \tag{3.14}$$

Proof. According to Definition 3.7, expression (3.11) states that

$$(A - \xi\,\hat{p}) \;\in\; b\,\partial\|\cdot\|_F(\hat{p}) \tag{3.15}$$

with $\partial\|\cdot\|_F$ denoting the subgradient of the Frobenius norm, where only trace-free arguments are under consideration. The Frobenius norm $\|\cdot\|_F : Q \to \mathbb{R}$ is a convex functional and so is (3.12). Identity (3.15) is equivalent to 0 belonging to the subgradient of (3.12), which characterizes the minimizers of (3.12). Moreover, there holds $\langle A, q\rangle_Q = \langle \operatorname{dev} A, q\rangle_Q$ for all trace-free elements $q \in Q$, whence the matrix A can be replaced by the matrix $\operatorname{dev} A$ in (3.11) and (3.12).

Let the domain Ω be separated into the three disjoint subsets

$$\Omega^e := \{x \in \Omega \mid \exists \text{ open } \omega \subset \Omega : \ x \in \omega \wedge \|\operatorname{dev} A\|_F - b \leq 0 \text{ in } \omega\}\,,$$
$$\Omega^p := \Omega \setminus \overline{\Omega^e}\,, \quad \text{and} \quad \Gamma^{ep} := \Omega \setminus (\Omega^e \cup \Omega^p)\,.$$

Note that Ω^e and Ω^p are open, that Γ^{ep} has measure zero, and that there holds $\|\operatorname{dev} A\|_F - b \leq 0$ on Ω^e and $\|\operatorname{dev} A\|_F - b > 0$ on Ω^p. Consequently, the minimization of (3.12) results in finding $\hat{p} \in Q$ with $\|\operatorname{tr} p\|_{L_2(\Omega)} = 0$, such that the functionals

$$J_i(\hat{p}) := \frac{\xi}{2}\int_{\Omega^i} \|\hat{p}\|_F^2 \,\mathrm{d}x - \int_{\Omega^i} \langle \operatorname{dev} A\,,\, \hat{p}\rangle_F \,\mathrm{d}x + \int_{\Omega^i} b\,\|\hat{p}\|_F \,\mathrm{d}x \quad i \in \{e,p\} \tag{3.16}$$

are minimized, or equivalently the inequalities

$$\int_{\Omega^i} \langle \operatorname{dev} A - \xi\hat{p}\,,\, q - \hat{p}\rangle_F \,\mathrm{d}x \leq \int_{\Omega^i} b\left(\|q\|_F - \|\hat{p}\|_F\right)\,\mathrm{d}x \quad i \in \{e,p\} \tag{3.17}$$

are satisfied for all $q \in Q$ with $\|\operatorname{tr} q\|_{L_2(\Omega)} = 0$.

We show identity (3.13). An application of the pointwise Cauchy-Schwarz inequality $\langle \operatorname{dev} A\,,\, \hat{p}\rangle_F \leq \|\operatorname{dev} A\|_F \|\hat{p}\|_F$ yields

$$J_e(\hat{p}) \geq \frac{\xi}{2}\int_{\Omega^e} \|\hat{p}\|_F^2 \,\mathrm{d}x + \int_{\Omega^e} \underbrace{(b - \|\operatorname{dev} A\|_F)}_{\geq 0} \|\hat{p}\|_F \,\mathrm{d}x \geq 0.$$

By choosing $\hat{p} = 0$ on Ω^e we obtain $J_e(\hat{p}) = 0$. Therefore,

$$\hat{p} = 0 \qquad \text{on } \Omega^e \tag{3.18}$$

minimizes J_e in (3.16). Moreover, there holds $\hat{p}(x) \neq 0$ on Ω^p which we show now by contradiction. Choose $\Omega' \subset \Omega^p$ arbitrary and fix. Assuming, that $\hat{p} = 0$ on Ω' and plugging it into (3.17) for $i = p$ would yield

$$\int_{\Omega'} \langle \operatorname{dev} A\,,\, q\rangle_F \,\mathrm{d}x \leq \int_{\Omega'} b\|q\|_F \,\mathrm{d}x$$

for all trace-free elements $q \in Q$, which satisfy $q = \hat{p}$ on $\Omega^p \setminus \Omega'$. By the choice $q = \operatorname{dev} A$ on Ω', we obtain $\int_{\Omega'} \|\operatorname{dev} A\|_F - b \,\mathrm{d}x \leq 0$, violating the definition of Ω^p. Thus there holds $\hat{p}(x) \neq 0$, and consequently $\partial\|\cdot\|_F(\hat{p}) = \{\hat{p}/\|\hat{p}\|_F\}$, on Ω^p. Hence, (3.15) reads

$$\operatorname{dev} A - \xi\,\hat{p} = b\,\frac{\hat{p}}{\|\hat{p}\|_F} \qquad \text{on } \Omega^p\,, \tag{3.19}$$

whence we conclude

$$\|\hat{p}\|_F = \frac{1}{\xi}\left(\|\operatorname{dev} A\|_F - b\right)\,. \tag{3.20}$$

Plugging (3.20) into (3.19) yields

$$\hat{p} = \frac{1}{\xi}\left(\|\operatorname{dev} A\|_F - b\right)\frac{\operatorname{dev} A}{\|\operatorname{dev} A\|_F} \text{ on } \Omega^p. \tag{3.21}$$

Combining the formulas (3.18) and (3.21) we obtain (3.13). Finally, plugging (3.13) into (3.12) yields (3.14). $\square$

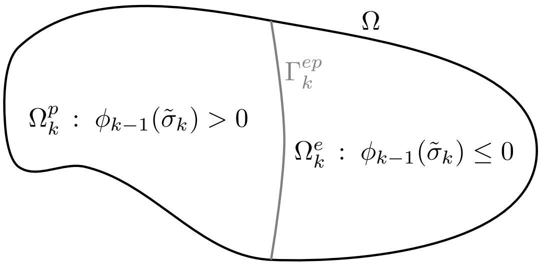

FIG. 3.1. *The trial stress $\tilde\sigma_k(\varepsilon(v))(x)$ rules the domain splitting of Ω at the kth time step via the yield functional ϕ_{k-1}, cf.* (3.22).

We define the trial stress $\tilde\sigma_k : Q \to Q$ at the kth time step and the yield function $\phi_{k-1} : Q \to \mathbb{R}$ (cf. (2.9)) at the $k-1$st time step by

$$\tilde\sigma_k(q) := \mathbb{C}(q - p_{k-1}) \quad \text{and} \quad \phi_{k-1}(\sigma) := \|\operatorname{dev}\sigma\|_F - \sigma_y(1 + H\,\alpha_{k-1})\,. \tag{3.22}$$

After using the substitution $\Delta\tilde p_k(\varepsilon(v)) = \tilde p_k(\varepsilon(v)) - p_{k-1}$, Theorem 3.8 says, that for a fixed displacement $v \in V_D$ the plastic strain minimizer $\tilde p_k(\varepsilon(v))$ of (3.4) reads

$$\tilde p_k(\varepsilon(v)) = \frac{1}{2\mu + \sigma_y^2 H^2} \max\{0, \phi_{k-1}(\tilde\sigma_k(\varepsilon(v)))\} \frac{\operatorname{dev}\tilde\sigma_k(\varepsilon(v))}{\|\operatorname{dev}\tilde\sigma_k(\varepsilon(v))\|_F} + p_{k-1}\,. \tag{3.23}$$

Therefore, if the minimizer $u_k \in V_D$ of the functional $J_k(\cdot) = \bar J_k(\cdot, \tilde p_k(\varepsilon(\cdot)))$ in (3.8) is known, then the plastic strain p_k at the time step k is provided by the formula (3.23) as $p_k = \tilde p_k(\varepsilon(u_k))$. Notice that the formula (3.23) also satisfies the necessary condition $\operatorname{tr} p_k = \operatorname{tr} p_{k-1}$ to guarantee the minimization property $J_k(u_k) = \bar J_k(u_k, p_k) < +\infty$ (cf. (3.4) and (3.2)).

At each time step k the domain Ω can be decomposed into three disjoint parts (see Figure 3.1), analogously to the decomposition we used in the proof to Theorem 3.8:

(i) $\Omega_k^e(v) := \{x \in \Omega \mid \exists \text{ open } \omega \subset \Omega :\ x \in \omega \wedge \phi_{k-1}(\tilde\sigma_k(\varepsilon(v))) \le 0 \text{ a. e. in } \omega\}$,
(ii) the set of plastic increment points $\Omega_k^p(v) := \Omega \setminus \overline{\Omega_k^e(v)}$,
(iii) and the set of elastoplastic interface points $\Gamma_k^{ep}(v) := \Omega \setminus (\Omega_k^p(v) \cup \Omega_k^e(v))$.

Obviously, both sets $\Omega_k^e(v)$ and $\Omega_k^p(v)$ are open, $\Gamma_k^{ep}(v)$ has zero measure, and that

$$\begin{aligned} \phi_{k-1}(\tilde\sigma_k(\varepsilon(v))) &\le 0 \quad \text{a. e. in } \Omega_k^e(v),\\ \phi_{k-1}(\tilde\sigma_k(\varepsilon(v))) &> 0 \quad \text{a. e. in } \Omega_k^p(v). \end{aligned} \tag{3.24}$$

For a one-time-step problem, the sets $\Omega^e(v) := \Omega_1^e(v)$ and $\Omega^p(v) := \Omega_1^p(v)$ specify elastically and plastically deformed parts of the continuum, respectively.

We obtain a smooth problem with respect to the displacement field u_k only:

PROBLEM 3.9. *Let $k \in \{1, \dots, N_\Theta\}$ denote the time step. Let $p_{k-1} \in Q$ and $\alpha_{k-1} \in L_2(\Omega)$ be given such, that $\alpha_{k-1} \ge 0$ holds almost everywhere. Find $u_k \in V_D$ such, that for all $v \in V_D$ there holds $J_k(u_k) \le J_k(v)$ with the strictly convex and Fréchet differentiable functional J_k defined in* (3.8) *using $\tilde p_k$ as in* (3.23)*. The Gâteaux differential of J_k is presented in* (3.9).

REMARK 3.10. There exists a unique solution u_k to Problem 3.9, since there exists a unique solution (u_k, p_k) to Problem 2.3, and since

$$J_k(u_k) = \bar J_k(u_k, \tilde p_k(\varepsilon(u_k))) = \bar J_k(u_k, p_k) \le \bar J_k(v, \tilde p_k(\varepsilon(v))) = J_k(v) \quad \forall\, v \in V_D\,.$$

4. Computing the Solution of the Smooth Problem. The minimizer $\tilde{p}_k$ in (3.23) is a continuous mapping of Q into Q. Thus, $D\,J_k(v\,;w)$ in (3.9) is continuous with respect to v as well, and a gradient method could be used for a numerical solution. Instead, we investigate the existence of the second derivative of $J_k(v)$ having in mind to use Newton's method.

4.1. An Attempt to Calculate the Second Derivative of J_k. The Gâteaux differential of $D\,J_k$ defined in (3.9) reads

$$D^2 J_k(v\,;\,w_1,w_2) = \langle \varepsilon(w_1) - D\,\tilde{p}_k(\varepsilon(v)\,;\,\varepsilon(w_1))\,,\,\varepsilon(w_2)\rangle_{\mathbb{C}} \quad \forall\, w_1,w_2 \in V_0$$

provided that the Gâteaux differential $D\,\tilde{p}_k(\varepsilon(v)\,;\,\varepsilon(w_1)) \in Q$ of the plastic strain minimizer $\tilde{p}_k(\varepsilon(v))$ defined in (3.23) exists in the whole domain Ω.

In the set $\Omega_k^e(v)$, where $\phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v))) \le 0$ (cf. (3.22)), there obviously holds

$$D\,\tilde{p}_k(\varepsilon(v)\,;\,q) = 0 \tag{4.1}$$

for all $q \in Q$, and therefore we obtain the formula known from theory of elasticity

$$D^2 J_k(v\,;\,w_1,w_2) = \langle \varepsilon(w_1)\,,\,\varepsilon(w_2)\rangle_{\mathbb{C}} \quad \forall\, w_1,w_2 \in V_0.$$

In the set of plastic increment points $\Omega_k^p(v)$, where $\phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v))) > 0$ holds a. e., the plastic strain reads

$$\tilde{p}_k(\varepsilon) = (2\mu + \sigma_y^2 H^2)^{-1} \phi_{k-1}(\tilde{\sigma}_k(\varepsilon))\, \frac{\operatorname{dev} \tilde{\sigma}_k(\varepsilon)}{\|\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\|_F}\,.$$

For the moment, we omit the dependency of ε on v in our notation, and calculate the Gâteaux differential of $\tilde{p}_k$ with respect to ε. By using the product and the chain rules, we obtain

$$\begin{aligned} D\,\tilde{p}_k(\varepsilon\,;\,q) = (2\mu + \sigma_y^2 H^2)^{-1} \Bigg(& D\,\phi_{k-1}(\tilde{\sigma}_k(\varepsilon)\,;\,D\,\tilde{\sigma}_k(\varepsilon\,;\,q)) \frac{\operatorname{dev} \tilde{\sigma}_k(\varepsilon)}{\|\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\|_F} \\ & + \phi_{k-1}(\tilde{\sigma}_k(\varepsilon)) D\, \frac{(\cdot)}{\|\cdot\|_F} (\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\,;\,D \operatorname{dev} \tilde{\sigma}_k(\varepsilon\,;\,q)) \Bigg)\,. \end{aligned}$$

Using the derivatives rules (cf. (3.22))

$$D\,\tilde{\sigma}_k(\varepsilon\,;\,q) = D\,\tilde{\sigma}_k(\,q) = \mathbb{C}\,q, \quad D \operatorname{dev} \tilde{\sigma}_k(\varepsilon\,;\,q) = D \operatorname{dev} \tilde{\sigma}_k(\,q) = 2\mu \operatorname{dev} q$$

and

$$D\,\phi_{k-1}(\sigma\,;\,\tau) = \frac{\langle \operatorname{dev}\sigma\,,\,D \operatorname{dev}(\sigma\,;\,\tau)\rangle_F}{\|\sigma\|_F}, \quad D\,\frac{(\cdot)}{\|\cdot\|_F}(\sigma\,;\,\tau) = \frac{\tau}{\|\sigma\|_F} - \frac{\sigma\langle\sigma\,,\,\tau\rangle_F}{\|\sigma\|_F^3},$$

we end up with the formula

$$\begin{aligned} D\,\tilde{p}_k(\varepsilon\,;\,q) = \frac{2\mu}{2\mu + \sigma_y^2 H^2} \Bigg(& \frac{\phi_{k-1}(\varepsilon)}{\|\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\|_F} \operatorname{dev} q \\ & + \left(1 - \frac{\phi_{k-1}(\varepsilon)}{\|\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\|_F}\right) \frac{\langle \operatorname{dev} \tilde{\sigma}_k(\varepsilon)\,,\,\operatorname{dev} q\rangle_F}{\|\operatorname{dev} \tilde{\sigma}_k(\varepsilon)\|_F^2} \operatorname{dev} \tilde{\sigma}_k(\varepsilon) \Bigg)\,. \end{aligned} \tag{4.2}$$

Unfortunately, $\tilde{p}_k$ in (3.23) is not differentiable on the set of elastoplastic interface points, $\Gamma_k^{ep}(v)$, due to the term $\max\{0,\phi_{k-1}\}$. To summarize it, the derivative $D\,\tilde{p}_k$ exists everywhere in the sets of elastic and plastic increment points, but is not computable on the elastoplastic interface (see Figure 3.1). Thus, $D^2 J_k(v)$ does not exist. No matter that the elastoplastic interface is a set of measure zero, a classical Newton method is not applicable to Problem 3.9.

4.2. Concept of Slant Differentiability. Our goal is to solve Problem 3.9 by means of a Newton-like method which replaces the requirement of the second derivative $D^2 J_k(v)$ in a way that the super-linear convergence rate still can be shown.

The main tool here to overcome the non-differentiability of $D\,J_k$ due to the mapping $\max\{0,\cdot\}$ is the concept of *slant differentiability*, which was introduced by X. Chen, Z. Nashed and L. Qi in [7]. Other concepts of semismoothness, e. g. [29, 30], or the smoothing (regularization) of non-differentiable terms, e. g. [22], are not outlined in this work, but might be considered as an alternative. The most important relations between smoothing and semismooth methods are discussed in [7, § 3].

Henceforth, let X, Y, and Z be Banach spaces, and $\mathcal{L}(\circ,\diamond)$ denote the set of all linear mappings of the set $\circ$ into the set $\diamond$.

DEFINITION 4.1. Let $U \subseteq X$ be an open subset. A function $F : U \to Y$ is said to be *slantly differentiable at* $x \in U$ if there exist

1. mappings $F^o : U \to \mathcal{L}(X,Y)$ and $r : X \to Y$ with $\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0$ which satisfy
$$F(x+h) = F(x) + F^o(x+h)\,h + r(h)$$
for all $h \in X$, for which $(x+h) \in U$, and
2. constants $\delta > 0$ and $C > 0$ such, that for all $h \in X$ with $\|h\| < \delta$ there holds
$$\|F^o(x+h)\|_{\mathcal{L}(X,Y)} := \sup_{y\in X\setminus\{0\}} \frac{\|F^o(x+h)\,y\|}{\|y\|} \le C\,.$$

We say, that $F^o(x)$ is a *slanting function for F at x.*

DEFINITION 4.2. Let $U \subseteq X$ be an open subset. A function $F : U \to Y$ is said to be *slantly differentiable in* U if there exists $F^o : U \to \mathcal{L}(X,Y)$ such that F^o is a slanting function for F at every point $x \in U$. F^o is said to be a *slanting function for F in U.*

REMARK 4.3. Similar to the relation between Gâteaux differential and Gâteaux derivative, we define the *slanting differential for F^o at x along the direction h* by $\tilde{F}^o : U \times X \to Y$ with $\tilde{F}^o(x\,;\,h) := F^o(x)h$. Since the mappings F^o and $\tilde{F}^o$ are taking a different number of arguments, it suffices to denote both by the same symbol F^o, i. e. $F^o(\cdot)$ for a slanting function and $F^o(\circ\,;\,\diamond)$ for the appropriate slanting differential.

REMARK 4.4. Both, the chain rule and the product rule hold for slanting functions in the same way as we know for classical derivatives.

THEOREM 4.5. *Let $U \subseteq X$ be an open subset, $F : U \to Y$ be a slantly differentiable function, and $F^o : U \to \mathcal{L}(X,Y)$ be a slanting function for F in U. Let $x^* \in U$ be a solution to the nonlinear problem $F(x) = 0$. If $F^o(x)$ is bijective in $\mathcal{L}(X,Y)$ for all $x \in U$, and if $\{\|F^o(x)^{-1}\| : x \in U\}$ is bounded, then the Newton-like iteration*

$$x^{j+1} = x^j - F^o(x^j)^{-1}F(x^j)\,, \qquad j \in \{0,1,2,\ldots\} \tag{4.3}$$

converges super-linearly to x^*, provided that $\|x^0 - x^*\|$ is sufficiently small.

Proof. See [7, Theorem 3.4]. □

We call a Newton-like method described by (4.3) a slant Newton method. The goal is to solve Problem 3.9 by finding $u_k \in V_D$ such, that $D\,J_k(u_k; w) = 0$ for all $w \in V_0$ with $D\,J_k$ as in (3.9). Therefore, we use the slant Newton method and choose

$$X = V\,,\quad Y = {V_0}^*\,,\quad U = V_D\,,\quad F = D\,J_k\,,\quad x^j = v^j\,,\quad \text{and}\quad x^* = u_k\,.$$

For each iteration step j we then have to solve:

$$\text{find } v^{j+1} \text{ in } V_D : \quad (D\,J_k)^o\,(v^j\,;\,v^{j+1} - v^j\,,\,w) = -D\,J_k(v^j\,;\,w) \quad \forall\, w \in V_0\,. \tag{4.4}$$

4.3. Slanting Functions for $\tilde{p}_k$ and $D\,J_k$. Henceforth we will use the following property, which is easy to verify: A Fréchet differentiable function is slantly differentiable, with the Fréchet derivative serving as a slanting function, and the Gâteaux differential serving as a slanting differential. Due to the chain rule we obtain

$$(D\,J_k)^o\,(v\,;\,w_1,w_2) = \langle \varepsilon(w_1) - \tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w_1))\,,\,\varepsilon(w_2)\rangle_{\mathbb{C}} \quad \forall\, w_1,w_2 \in V_0\,. \tag{4.5}$$

It remains to calculate the slanting function $\tilde{p_k}^o$. Taking to account, that a Fréchet derivative serves as a slanting function, we obtain from (4.1) and (4.2), that

$$\tilde{p_k}^o(\varepsilon(v)\,;\,q) = \begin{cases} 0 & \text{in } \Omega_k^e(v)\,, \\ \xi\left(\beta_k \operatorname{dev} q + (1-\beta_k)\,\frac{\langle \operatorname{dev}\tilde{\sigma}_k\,,\,\operatorname{dev} q\rangle_F}{\|\operatorname{dev}\tilde{\sigma}_k\|_F^2} \operatorname{dev}\tilde{\sigma}_k\right) & \text{in } \Omega_k^p(v)\,, \end{cases}$$

where the abbreviations

$$\xi := \frac{2\mu}{2\mu + \sigma_y^2 H^2}, \quad \beta_k := \frac{\phi_{k-1}(\tilde{\sigma}_k)}{\|\operatorname{dev}\tilde{\sigma}_k\|_F}, \quad \tilde{\sigma}_k := \tilde{\sigma}_k(\varepsilon(v)) \tag{4.6}$$

with the mappings ϕ_{k-1} and $\tilde{\sigma}_k$ defined in (3.22) are used. Notice, hence the modulus of hardening H, the yield stress σ_y, and the Lamé parameter μ are positive and due to (2.11), (3.22) and (3.24), we always have

$$\xi \in \,]0,1[\quad \text{and} \quad \beta_k : \Omega_k^p(v) \to \,]0,1[\,. \tag{4.7}$$

The minimizer $\tilde{p}_k$ is not differentiable on $\Gamma_k^{ep}(v)$ due to the term $\max\{0,\phi_k\}$ in (3.23).

M. Hintermüller, K. Ito and K. Kunisch discuss the slant differentiability of the mapping $\max\{0,y\}$ for certain Banach spaces, that is, for the finite dimensional case $y \in \mathbb{R}^n$ in [17, Lemma 3.1], and the infinite dimensional case $y \in L_q(\Omega)$ in [17, Proposition 4.1]. Let us summarize their results in the following two theorems.

Theorem 4.6 (The finite dimensional case). *Let $n \in N$ be arbitrary, and F be a mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ defined as $F(y) := \max\{0,y\}$. Then, F is slantly differentiable, and for $\gamma \in \mathbb{R}^n$ fixed arbitrarily, the matrix valued function*

$$F^o(y) := \operatorname{diag}\left(f_i(y_i)\right)_{i=1}^n \quad \textit{with} \quad f_i(z) = \begin{cases} 0 & \textit{if } z < 0\,, \\ 1 & \textit{if } z > 0\,, \\ \gamma_i & \textit{if } z = 0 \end{cases} \tag{4.8}$$

serves as a slanting function.

The next theorem addresses the slant differentiability of the mapping $\max\{0,y\}$ in the infinite dimensional case $y \in L_q(\Omega)$. Therefore, we define a decomposition of the domain Ω into subdomains $\Omega = \Omega_{\le} \cup \Gamma_| \cup \Omega_>$ such, that analogously to Figure 3.1 the interface $\Gamma_|$ separates $\Omega_>$ ($y > 0$ a. e.) from $\Omega_{\le}$ ($y \le 0$ a. e.).

Theorem 4.7 (The infinite dimensional case). *Let p and q in $\mathbb{R}$ be fixed arbitrarily such that $1 \le p \le q \le +\infty$ is satisfied, and let F be a mapping of $L_q(\Omega)$ into $L_p(\Omega)$ defined as $F(y) := \max\{0,y\}$. Then, for γ fixed arbitrarily in $\mathbb{R}$, the function*

$$F^o(y)(x) := \begin{cases} 0 & \textit{on } \Omega_{\le}\,, \\ 1 & \textit{on } \Omega_>\,, \\ \gamma & \textit{on } \Gamma_| \end{cases} \tag{4.9}$$

serves as a slanting function for F if $p < q$, but F^o does in general not serve as a slanting function for F if $p = q$.

We apply the last two theorems to find a slanting function for the minimizer $\tilde{p}_k(\varepsilon)$ defined in (3.23), each in the spatially continuous and discrete case. The task turns out to be trivial in the latter case (see § 4.5), but some further regularity assumptions are required in the spatially continuous case due to the following considerations:

The minimizer $\tilde{p}_k$ works as a mapping from Q into Q to keep the energy functional J_k in (3.8) well-defined. The explicit formula (3.23) says, that $\tilde{p}_k$ maps into Q only if

$$\max\{0, \phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v)))\}$$

maps into $L_2(\Omega)$, where ϕ_{k-1} and $\tilde{\sigma}_k$ are defined in (3.22). By the application of Theorem 4.7 to the slant differentiation of the max-term measured in the $L_2(\Omega)$-norm we have to guarantee, that its argument

$$\phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v)))$$

is bounded in the $L_{2+\epsilon}(\Omega)$-norm for some $\epsilon > 0$ and for all $v \in V_D$, or at least for those $v \in V_D$ which are run through by the slant Newton method (4.4). This issue is not further discussed in this work, but left as an open question for more theoretical analysis on regularity results in elastoplastic problems.

Under the assumption

$$\phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v))) \in L_{2+\epsilon}(\Omega)\,,$$

we can formulate an immediate result from the combination of the product rule, chain rule, Theorem 4.7 (where we choose $\gamma = 0$), and the results from § 4.1.

Corollary 4.8. *Let $k \in \{1, \dots, N_\Theta\}$ and $v \in V_D$ be arbitrarily fixed. If there exists $\epsilon > 0$ such that $\phi_{k-1}(\tilde{\sigma}_k(\varepsilon(v)))$, as defined in* (3.22)*, is in $L_{2+\epsilon}(\Omega)$, then the mapping $\tilde{p}_k : Q \to Q$ defined in* (3.23) *is slantly differentiable at $\varepsilon(v)$. The mapping*

$$\tilde{p_k}^o(\varepsilon(v)\,;\,q) = \begin{cases} \xi\left(\beta_k \operatorname{dev} q + (1-\beta_k)\ \frac{\langle \operatorname{dev}\tilde{\sigma}_k\,,\operatorname{dev} q\rangle_F}{\|\operatorname{dev}\tilde{\sigma}_k\|_F^2}\operatorname{dev}\tilde{\sigma}_k\right) & \text{in } \Omega_k^p(v)\,, \\ 0 & \text{else}\,, \end{cases} \tag{4.10}$$

for all $q \in Q$ serves as a slanting function for $\tilde{p}_k$ at $\varepsilon(v)$, wherein the abbreviations (4.6) *together with the definitions* (3.22) *are used. Moreover, the functional $D\,J_k(v)$ is slantly differentiable with the slanting function $(D\,J_k)^o\,(v)$ as in* (4.5)*.*

Corollary 4.8 corresponds to Corollary 4.13 in § 4.5 on page 16, which states the slant differentiability of the energy functional's first derivative $D\,J_k$ and the plastic strain minimizer $\tilde{p}_k$ in finite dimensional FE-spaces. Unlike the infinite dimensional case, no additional assumptions will be necessary in the finite dimensional case (cf. Theorem 4.6 and Theorem 4.7).

4.4. Local Super-Linear Convergence in Infinite Dimensions. The application of Theorem 4.5 requires the existence and uniform boundedness of the inverse map $[(D\,J_k)^o]^{-1}$. These properties are shown in Proposition 4.11 on page 15 by using, that the bilinear form $(D\,J_k)^o\,(v) := (D\,J_k)^o\,(v\,;\diamond,\circ)$ is bounded and elliptic, which we show now in the following lemma.

Lemma 4.9. *Let $k \in \{1, \dots, N_\Theta\}$ and $v \in V_D$ be fixed arbitrarily. Let the mapping $(D\,J_k)^o : V_D \to \mathcal{L}(V_0, {V_0}^*)$ be defined $(D\,J_k)^o\,(v) := (D\,J_k)^o\,(v\,;\diamond,\circ)$ as in* (4.5) *with $\tilde{p_k}^o$ as in* (4.10)*. Then there exist positive constants κ_1 and κ_2 which satisfy*

$$(D\,J_k)^o\,(v\,;\,w,w) \ge \kappa_1 \|w\|_V^2 \quad \forall\, w \in V_0 \quad \textit{(ellipticity)}\,, \tag{4.11}$$

$$(D\,J_k)^o\,(v\,;\,w,\overline{w}) \le \kappa_2 \|w\|_V \|\overline{w}\|_V \quad \forall\, w, \overline{w} \in V_0 \quad \textit{(boundedness)}\,. \tag{4.12}$$

Proof. We show the ellipticity (4.11). By the definition of $(D\,J_k)^o$ in (4.5), there holds

$$(D\,J_k)^o\,(v\,;\,w,w) = \langle \varepsilon(w) - \tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\,,\,\varepsilon(w)\rangle_{\mathbb{C}}\,. \tag{4.13}$$

First, we prove the contractivity of the operator $p_k{}^o(\varepsilon(v),\cdot)$ defined in (4.10) with respect to its second argument:

$$\begin{aligned}
\|\tilde{p_k}^o(\varepsilon(v)\,;\,q)\|_{\mathbb{C}}^2 &= \int_\Omega \langle \mathbb{C}p_k{}^o(\varepsilon(v)\,;\,q)\,,\,p_k{}^o(\varepsilon(v)\,;\,q)\rangle_F \,\mathrm{d}x = 2\mu \int_\Omega \|p_k{}^o(\varepsilon(v)\,;\,q)\|_F^2 \,\mathrm{d}x \\
&= \xi^2\, 2\mu \int_{\Omega_k^p(v)} \|\beta_k \mathrm{dev}\, q + (1-\beta_k)\, \frac{\langle \mathrm{dev}\,\tilde{\sigma}_k\,,\,\mathrm{dev}\, q\rangle_F}{\|\mathrm{dev}\,\tilde{\sigma}_k\|_F^2} \mathrm{dev}\,\tilde{\sigma}_k\|_F^2 \,\mathrm{d}x \\
&\le \xi^2\, 2\mu \int_\Omega \|\mathrm{dev}\, q\|_F^2 = \xi^2 \int_\Omega \langle \mathbb{C}\mathrm{dev}\, q\,,\,\mathrm{dev}\, q\rangle_F \,\mathrm{d}x \\
&\le \xi^2 \int_\Omega \langle \mathbb{C}q\,,\,q\rangle_F \,\mathrm{d}x = \xi^2 \|q\|_{\mathbb{C}}^2 \quad \forall\, q \in Q.
\end{aligned}$$

Then the substitution of this estimate to (4.13) yields

$$(D\,J_k)^o\,(v\,;\,w,w) \ge (1-\xi)\|\varepsilon(w)\|_{\mathbb{C}}^2\,,$$

which together with Korn's inequality from the theory of linear elasticity (there exists a constant $\kappa_1^{\mathrm{e}} > 0$ such, that $\|\varepsilon(w_1)\|_{\mathbb{C}}^2 \ge \kappa_1^{\mathrm{e}} \|w_1\|_V^2$ holds for all w_1 in V_0) already provides the ellipticity with the constant $\kappa_1 := (1-\xi)\,\kappa_1^{\mathrm{e}}$.

We show the boundedness (4.12). The Cauchy-Schwarz inequality yields

$$(D\,J_k)^o\,(v\,;\,w,\overline{w}) \le \|\varepsilon(w) - \tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_{\mathbb{C}} \|\varepsilon(\overline{w})\|_{\mathbb{C}} \quad \forall\, w,\overline{w} \in V_0\,. \tag{4.14}$$

Let $w,\overline{w} \in V_0$ be fixed arbitrarily. Since $\|\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_F = 0$ a. e. in $\Omega_k^e(v)$, and

$$\begin{aligned}
\|\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_F^2 &= \xi^2 \left(\beta_k^2 \|\mathrm{dev}\,\varepsilon(w)\|_F^2 + (1+\beta_k)(1-\beta_k) \frac{\langle \mathrm{dev}\,\tilde{\sigma}_k\,,\,\mathrm{dev}\,\varepsilon(w)\rangle_F^2}{\|\mathrm{dev}\,\tilde{\sigma}_k\|_F^2} \right) \\
&\le \xi \left(\beta_k \|\mathrm{dev}\,\varepsilon(w)\|_F^2 + 2(1-\beta_k) \frac{\langle \mathrm{dev}\,\tilde{\sigma}_k\,,\,\mathrm{dev}\,\varepsilon(w)\rangle_F^2}{\|\mathrm{dev}\,\tilde{\sigma}_k\|_F^2} \right) \\
&\le 2\xi \left(\beta_k \|\mathrm{dev}\,\varepsilon(w)\|_F^2 + (1-\beta_k) \frac{\langle \mathrm{dev}\,\tilde{\sigma}_k\,,\,\mathrm{dev}\,\varepsilon(w)\rangle_F^2}{\|\mathrm{dev}\,\tilde{\sigma}_k\|_F^2} \right) \\
&= 2\langle \mathrm{dev}\,\varepsilon(w)\,,\,\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\rangle_F
\end{aligned}$$

a. e. in $\Omega_k^p(v)$, there holds $\|\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_Q^2 \le 2\langle \mathrm{dev}\,\varepsilon(w)\,,\,\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\rangle_Q$, from which we obtain by elementary calculation

$$\|\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_{\mathbb{C}}^2 \le 2\langle \varepsilon(w)\,,\,\tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\rangle_{\mathbb{C}}\,. \tag{4.15}$$

Due to (4.15), there holds $\|\varepsilon(w) - \tilde{p_k}^o(\varepsilon(v)\,;\,\varepsilon(w))\|_{\mathbb{C}}^2 \le \|\varepsilon(w)\|_{\mathbb{C}}^2$, which applied to the inequality (4.14) yields $(D\,J_k)^o\,(v\,;\,w,\overline{w}) \le \|\varepsilon(w)\|_{\mathbb{C}} \|\varepsilon(\overline{w})\|_{\mathbb{C}}$. Again, from the theory of elasticity it is well known, that there exists a constant $\kappa_2^{\mathrm{e}} > 0$ such, that $\langle \varepsilon(w_1)\,,\,\varepsilon(w_2)\rangle_{\mathbb{C}} \le \kappa_2^{\mathrm{e}} \|w_1\|_V \|w_2\|_V$ holds for all w_1 and w_2 in V_0. Thus, the boundedness (4.12) holds with $\kappa_2 := \kappa_2^{\mathrm{e}}$. Notice, that κ_1 and κ_2 are independent from the certain choice of $v \in V_D$. $\square$

REMARK 4.10. The ellipticity constant reads $\kappa_1 = (1-\xi)\,\kappa_1^e$, where ξ is defined in (4.6) as $\xi = 2\mu/\left(2\mu + \sigma_y^2 H^2\right)$ and κ_1^e is the ellipticity constant of the linear elastic bilinear form $\langle \varepsilon(\circ)\,,\,\varepsilon(\diamond)\rangle_{\mathbb{C}}$. Therefore, the ellipticity property of $(D\,J_k)^o\,(v\,;\,\circ,\diamond)$ gets lost if ξ goes to 1, which is the case if the yield stress σ_y or the modulus of hardening H approach zero. Hence the ellipticity property is essential for what comes next, this work may not be extended to the perfect plastic case $H = 0$ straightforward.

THEOREM 4.11. *Let $k \in \{1,\ldots,N_\Theta\}$ be fixed and the assumptions of Corollary* 4.8 *be fulfilled. Let the mapping $D\,J_k : V_D \to {V_0}^*$ be defined $D\,J_k(v) := D\,J_k(v\,;\,\circ)$ as in* (3.9)*, and $(D\,J_k)^o : V_D \to \mathcal{L}(V_0, {V_0}^*)$ be defined $(D\,J_k)^o\,(v) := (D\,J_k)^o\,(v\,;\,\diamond,\circ)$ as in* (4.5)*. Then, the sequence of the slant Newton iterates*

$$v^{j+1} = v^j - \left[(D\,J_k)^o\,(v^j)\right]^{-1} D\,J_k(v^j)$$

converges super-linearly to the solution u_k of Problem 3.9*, provided that $\|v^0 - u_k\|_V$ is sufficiently small.*

Proof. We check the assumptions of Theorem 4.5 for the choice $F = D\,J_k$. Let $v \in V_D$ be arbitrarily fixed. The mapping $(D\,J_k)^o\,(v) : V_0 \to {V_0}^*$ serves as a slanting function for $D\,J_k$ at v. Moreover, $(D\,J_k)^o\,(v) : V_0 \to {V_0}^*$ is bijective if and only if there exists a unique element w in V_0 such, that for arbitrary but fixed $f \in {V_0}^*$ there holds

$$(D\,J_k)^o\,(v\,;\,w,\overline{w}) = f(\overline{w}) \quad \forall\,\overline{w} \in V_0\,. \tag{4.16}$$

Since the bilinear form $(D\,J_k)^o\,(v)$ is elliptic and bounded (Lemma 4.9), we apply the Lax-Milgram Theorem to ensure the existence of a unique solution to (4.16). Finally, the uniform boundedness of $\left[(D\,J_k)^o\,(\cdot)\right]^{-1}$ follows from the estimate

$$\begin{aligned}
\|\left[(D\,J_k)^o\,(v)\right]^{-1}\| &= \sup_{w^*\in {V_0}^*} \frac{\|\left[(D\,J_k)^o\,(v)\right]^{-1}\,w^*\|_V}{\|w^*\|_{{V_0}^*}} = \sup_{w\in V_0} \frac{\|w\|_V}{\|(D\,J_k)^o\,(v\,;\,w,\cdot)\|_{{V_0}^*}} \\
&= \sup_{w\in V_0}\inf_{\overline{w}\in V_0} \frac{\|w\|_V\|\overline{w}\|_V}{|(D\,J_k)^o\,(v\,;\,w,\overline{w})|} \le \sup_{w\in V_0} \frac{\|w\|_V^2}{|(D\,J_k)^o\,(v\,;\,w,w)|} \le \frac{1}{\kappa_1}\,,
\end{aligned}$$

with κ_1 denoting the v-independent ellipticity constant of Lemma 4.9. ∎

4.5. Local Super-Linear Convergence in Finite Dimensions. Let $\mathcal{T}$ be a shape regular triangulation of Ω. We approximate the infinite dimensional spaces V, Q and $L_2(\Omega)$ by finite dimensional subspaces $V_h \subset V$, $Q_h \subset Q$ and $L_h \subset L_2(\Omega)$ such, that $\varepsilon(v_h) \in Q_h$ and $\|q_h\|_F \in L_h$ holds true for all $v_h \in V_h$ and $q_h \in Q_h$ in the weak sense, and $v_h \in C^1(T)$ for all $T \in \mathcal{T}$. We further define the finite dimensional hyper plains $V_{hD} := V_h \cap V_D$ and $V_{h0} := V_h \cap V_0$. Then, the finite dimensional problem corresponding to Problem 3.9 reads:

PROBLEM 4.12. *Let $k \in \{1,\ldots,N_\Theta\}$ denote the time step. Let $p_{h,k-1} \in Q_h$ and $\alpha_{h,k-1} \in L_{h2}(\Omega)$ be given such, that $\alpha_{h,k-1} \ge 0$ holds true almost everywhere in Ω. Find $u_{h,k} \in V_{hD}$ which satisfies*

$$D\,J_k(u_{h,k}) = 0\,.$$

Here, $D\,J_k : V_{hD} \to {V_{h0}}^*$ is defined by $D\,J_k(v_h) := D\,J_k(v_h\,;\,\circ)$ as in (3.9) with the mapping $\tilde{p}_k : Q_h \to Q_h$ defined as in (3.23). Analogous results to Corollary 4.8 and Theorem 4.11 can be shown for the finite dimensional subspace V_h without any additional assumptions:

COROLLARY 4.13. *Let $k \in \{1,\dots,N_\Theta\}$ and $v_h \in V_{hD}$ be arbitrarily fixed. Let $D\,J_k : (V_{hD}, V_{h0}^*) \to \mathbb{R}$ and $\tilde{p}_k : Q_h \to Q_h$ be defined as in* (3.9) *and* (3.23)*. Then, DJ_k is slantly differentiable at v_h and $\tilde{p}_k$ is slantly differentiable at $\varepsilon(v_h)$ with the slanting functions*

$$(D\,J_k)^o\,(v_h\,;\,w_h, \overline{w}_h) = \sum_{T\in\mathcal{T}} \int_T \mathbb{C}\,(\varepsilon(w_h) - \tilde{p_k}^o(\varepsilon(v_h)\,;\,\varepsilon(w_h))) : \varepsilon(\overline{w}_h)\,\mathrm{d}x\,, \tag{4.17}$$

$$\tilde{p_k}^o(\varepsilon(v_h)\,;\,q) = \begin{cases} \xi\left(\beta_k\,\mathrm{dev}\,q + (1-\beta_k)\,\frac{\langle \mathrm{dev}\,\tilde{\sigma}_k\,,\,\mathrm{dev}\,q\rangle_F}{\|\mathrm{dev}\,\tilde{\sigma}_k\|_F^2}\mathrm{dev}\,\tilde{\sigma}_k\right) & \text{in } \Omega_k^p(v_h)\,, \\ 0 & \text{else}\,, \end{cases}$$

for all $w_h, \overline{w}_h \in V_{h0}$. Herein the abbreviations $q = \varepsilon(w_h)$ and (4.6) *together with the definitions* (3.22) *are used.*

Proof. The result follows due to the piecewise continuously differentiability of v_h, which implies that $\mathrm{dev}\,\tilde{\sigma}_k(\varepsilon(v_h))$ and $\phi_{k-1}(\mathrm{dev}\,\tilde{\sigma}_k(\varepsilon(v_h)))$ in (3.22) are piecewise continuous mappings into $\mathbb{R}$, and thus Theorem 4.6, with $\gamma = 0$, is applicable. ∎

THEOREM 4.14. *Let $k \in \{1,\dots,N_\Theta\}$ denote a fixed time step. Let the mapping $D\,J_k : V_{hD} \to {V_{h0}}^*$ be defined by $D\,J_k(v_h) := D\,J_k(v_h\,;\,\circ)$ as in* (3.9)*, and let $(D\,J_k)^o : V_{hD} \to \mathcal{L}(V_{h0}, {V_{h0}}^*)$ be defined by $(D\,J_k)^o\,(v_h) := (D\,J_k)^o\,(v_h\,;\,\diamond,\circ)$ as in* (4.17)*. Then, the sequence of the slant Newton iterates*

$$v_h^{j+1} = v_h^j - \left[(D\,J_k)^o\,(v_h^j)\right]^{-1} D\,J_k(v_h^j)$$

converges super-linearly to the solution u_{hk} of Problem 4.12*, provided that $\|v_h^0 - u_{hk}\|_V$ is sufficiently small.*

Proof. Due to the subspace property $V_h \subset V$, this proof can be done analogously to the one for Theorem 4.11. ∎

5. Numerical Solution. This section is based on [1], where we consider the 2D case only. The major parts of an implementation in Matlab are outlined in [13].

We approximate the possibly non-polygonal 2D domain Ω by a polygonal 2D domain Ω' with the boundary $\Gamma' := \partial\Omega'$, which is split into the approximated Dirichlet and Neumann part Γ'_D and Γ'_N. Let $\mathcal{T} = \{T \text{ open} \subset \Omega'\}$ be a shape-regular triangulation of Ω', where all T are triangles, $\mathcal{E} = \{E\}$ be a set of edges and $\mathcal{E}_N = \mathcal{E} \cap \Gamma'_N$ be its intersection with the approximated Neumann boundary Γ'_N. The vertices of all triangles are collected in the set $\mathcal{N} = \{\underline{x} \in \mathbb{R}^2 \mid \exists\, T \in \mathcal{T} : \underline{x} \text{ is vertex of } T\}$. Let $V_h := \{v_h \in V \mid v_h \in \left[\mathcal{P}^1(T)\right]^3\ \forall\, T \in \mathcal{T}\}$, $Q_h := \{q_h \in Q \mid q_h \in \left[\mathcal{P}^0(T)\right]^{3\times 3}\ \forall\, T \in \mathcal{T}\}$ and $L_{h2}(\Omega) := \{\beta_h \in L_2(\Omega) \mid \beta_h \in \mathcal{P}^0(T)\ \forall\, T \in \mathcal{T}\}$, where $\mathcal{P}^n(X)$ denotes the set of all polynomials of order n defined on the set X. As a basis of V_h we choose piecewise linear nodal ansatz functions $\Phi(x) = (\phi_{i,j}(x))_{i\in\{1,\dots,\mathcal{N}\},j\in\{1,2\}}$. To each element $v_h \in V_h$, a vector $\underline{v} := (v_{h,j}(\underline{x}_i))_{i\in\{1,\dots,|\mathcal{N}|\},j\in\{1,2\}}$ can be associated using the identity

$$v_h(x) = \Phi(x)^T\,\underline{v}\,.$$

We consider the plain strain model, where $\varepsilon_{i3} = \varepsilon_{3i} = 0$ for all $i \in \{1,2,3\}$. The chosen structure of ε in the plain strain case implies a certain structure of the plastic strain p, caused by the minimizer formula (3.23), and of the stress σ, caused by Hooke's Law (2.4):

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}\,, \quad p = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}\,, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}\,.$$

Common (Tensor) Representation	Vector Representation
$\varepsilon := \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\underline{\varepsilon} := \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\,\varepsilon_{12} \end{bmatrix}$
$\sigma_\varepsilon := \mathbb{C}\,\varepsilon = \begin{bmatrix} \sigma_{\varepsilon,11} & \sigma_{\varepsilon,12} & 0 \\ \sigma_{\varepsilon,12} & \sigma_{\varepsilon,22} & 0 \\ 0 & 0 & \sigma_{\varepsilon,33} \end{bmatrix}$ with $\quad \mathbb{C}\,\varepsilon = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}\,\varepsilon\, I$	$\underline{\sigma}_\varepsilon := \begin{bmatrix} \sigma_{\varepsilon,11} \\ \sigma_{\varepsilon,22} \\ \sigma_{\varepsilon,12} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda + 2\,\mu & \lambda & 0 \\ \lambda & \lambda + 2\,\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}_{=:C} \underline{\varepsilon}\,,$ $\sigma_{\varepsilon,33} = \frac{\lambda}{\underbrace{2\,(\lambda+\mu)}_{=\nu}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \underline{\sigma}_\varepsilon, \;\; \mathrm{tr}\,\sigma_\varepsilon = \frac{\nu+1}{\nu}\,\sigma_{\varepsilon,33}$
$\mathrm{dev}\,\sigma_\varepsilon = \sigma_\varepsilon - \frac{\mathrm{tr}\,\sigma_\varepsilon}{\mathrm{tr}\,I}\, I$	$\underline{\mathrm{dev}\,\sigma}_\varepsilon := \begin{bmatrix} (\mathrm{dev}\,\sigma_\varepsilon)_{11} \\ (\mathrm{dev}\,\sigma_\varepsilon)_{22} \\ (\mathrm{dev}\,\sigma_\varepsilon)_{12} \end{bmatrix} = \underline{\sigma}_\varepsilon - \frac{\mathrm{tr}\,\sigma_\varepsilon}{\mathrm{tr}\,I} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$ thus, $\underline{\mathrm{dev}\,\sigma}_\varepsilon = \underbrace{\left(I - \frac{\nu+1}{\dim(\sigma_\varepsilon)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)}_{=:K} \underline{\sigma}_\varepsilon$
$p = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & -(p_{11}+p_{22}) \end{bmatrix}$	$\underline{p} := \begin{bmatrix} p_{11} \\ p_{22} \\ p_{12} \end{bmatrix},\; \lVert\underline{p}\rVert_N^2 := \underline{p}^T \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{=:N} \underline{p}\,,$ then: $\quad \lVert\underline{p}\rVert_N = \lVert p\rVert_F$
$\sigma_p := \mathbb{C}\,p = \begin{bmatrix} \sigma_{\varepsilon,11} & \sigma_{\varepsilon,12} & 0 \\ \sigma_{\varepsilon,12} & \sigma_{\varepsilon,22} & 0 \\ 0 & 0 & \sigma_{\varepsilon,33} \end{bmatrix}$ with $\quad \mathbb{C}\,p = 2\,\mu\,p + \lambda\,\underbrace{\mathrm{tr}\,p}_{=0}\,I = 2\,\mu\,p$	$\underline{\sigma}_p := \begin{bmatrix} \sigma_{p,11} \\ \sigma_{p,22} \\ \sigma_{p,12} \end{bmatrix} = 2\,\mu\,\underline{p}$ and $\quad \sigma_{p,33} = -\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \underline{\sigma}_p$
$\sigma = \mathbb{C}\,(\varepsilon - p) = \sigma_\varepsilon - \sigma_p$	$\underline{\sigma} = \underline{\sigma}_\varepsilon - \underline{\sigma}_p$ and $\sigma_{33} = \sigma_{\varepsilon,33} - \sigma_{p,33}$
$\mathrm{dev}\,\sigma = \mathrm{dev}\,\sigma_\varepsilon - \underbrace{\mathrm{dev}\,\sigma_p}_{=\sigma_p}\,,$ $\lVert\mathrm{dev}\,\sigma\rVert_F^2 = \sum_{i,j} (\mathrm{dev}\,\sigma)_{ij}^2$	$\underline{\mathrm{dev}\,\sigma} = \underline{\mathrm{dev}\,\sigma}_\varepsilon - \underline{\sigma}_p\,,\; \lVert\mathrm{dev}\,\sigma\rVert_F = \lVert\underline{\mathrm{dev}\,\sigma}\rVert_N\,,$ $(\mathrm{dev}\,\sigma)_{33} = -\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \underline{\mathrm{dev}\,\sigma}$

TABLE 5.1
Table of Vector Representation regarding the Plain Strain Model.

The information regarding ε, p and σ can be equivalently stored in the vectors

$$\underline{\varepsilon} := \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & 2\,\varepsilon_{12} \end{bmatrix}^T, \quad \underline{p} := \begin{bmatrix} p_{11} & p_{22} & p_{12} \end{bmatrix}^T, \quad \underline{\sigma} := \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{bmatrix}^T.$$

Corresponding operations in tensor and vector representation, such as norms, traces and deviators, are summarized in Table 5.1. Besides the results in Table 5.1, there holds $\langle \sigma_\varepsilon\,,\, \varepsilon\rangle_F = \underline{\sigma}_\varepsilon^T\,\underline{\varepsilon}$ and $\langle \sigma_p\,,\, \varepsilon\rangle_F = \underline{\sigma}_p^T\,\underline{\varepsilon}$.

Let R_T and R_E be restriction operators for the vector $\underline{u}$ to a local element T, i. e.

$$\underline{u}_T = R_T \underline{u}\,, \quad \underline{u}_E = R_E \underline{u}\,. \tag{5.1}$$

Let the fixed triangle $T \in \mathcal{T}$ have the vertices $\left(\underline{x}_\alpha, \underline{x}_\beta, \underline{x}_\gamma\right)$ with the coordinates

$$\left((x_{\alpha,1}, x_{\alpha,2})\,, (x_{\beta,1}, x_{\beta,2})\,, (x_{\gamma,1}, x_{\gamma,2})\right)\,.$$

Then $\underline{\varepsilon}(u_h)$ can be calculated on T by

$$\underline{\varepsilon}(u_h)(\underline{x})_{|T} = \begin{bmatrix} \partial_1\varphi_\alpha & 0 & \partial_1\varphi_\beta & 0 & \partial_1\varphi_\gamma & 0 \\ 0 & \partial_2\varphi_\alpha & 0 & \partial_2\varphi_\beta & 0 & \partial_2\varphi_\gamma \\ \partial_2\varphi_\alpha & \partial_1\varphi_\alpha & \partial_2\varphi_\beta & \partial_1\varphi_\beta & \partial_2\varphi_\gamma & \partial_1\varphi_\gamma \end{bmatrix} \begin{bmatrix} u_{\alpha,1} \\ u_{\alpha,2} \\ u_{\beta,1} \\ u_{\beta,2} \\ u_{\gamma,1} \\ u_{\gamma,2} \end{bmatrix},$$

or in a more compact way,

$$\underline{\varepsilon}(u_h)(x)_{|T} = B\,\underline{u}_T\,, \tag{5.2}$$

where the partial derivatives of φ_α, φ_β, and φ_γ can be obtained by

$$\nabla \begin{bmatrix} \varphi_\alpha \\ \varphi_\beta \\ \varphi_\gamma \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_{\alpha,1} & x_{\beta,1} & x_{\gamma,1} \\ x_{\alpha,2} & x_{\beta,2} & x_{\gamma,2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Integration over body and surface forces may be realized by the midpoint rule. We approximate f_k and g_k by $f_T := f_k(\overline{x}_T)$ and $g_E := g_k(\overline{x}_E)$, where $\overline{x}_T$ and $\overline{x}_E$ denote the center of mass of the element T, and the edge E, respectively. Defining

$$\underline{f}_T := \frac{|T|}{3}\, R_T^T f_T\,, \quad \text{and} \quad \underline{g}_E := \frac{|E|}{2}\, R_E^T g_E\,,$$

on each $T \in \mathcal{T}$ and on each $E \in \mathcal{E}_N$ there hold

$$\int_T f^T v_h \,\mathrm{d}x \approx \underline{f}_T^T \underline{v}\,, \quad \text{and} \quad \int_E g^T v_h \,\mathrm{d}s \approx \underline{g}_E^T \underline{v}\,. \tag{5.3}$$

5.1. Derivatives and Slanting Functions in Vector Representation. The whole integral over Ω can be split into a sum of integrals on single elements $T \in \mathcal{T}$. Therefore, by combining (5.1), (5.2) and (5.3) we obtain from (3.9) the discrete formulation of the energy functional's Gâteaux-differential

$$D\,J_k(\underline{u}\,;\,\underline{v}) := \sum_{T\in\mathcal{T}} \left(|T| \,\left(C\,B\,\underline{u}_T - 2\mu\,\underline{\tilde{p}}_k(B\,\underline{u}_T)\right)^T B\,R_T - \underline{f}_T^T \right) \underline{v} - \sum_{E\in\mathcal{E}_N} \underline{g}_E^T \underline{v}$$

with

$$\underline{\tilde{p}}_k(B\,\underline{u}_T) := \frac{\max\{0, \phi_{k-1}(\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T))\}}{2\mu + \sigma_y^2 H^2} \frac{\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T)}{\|\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T)\|_N} + \underline{p}_{k-1}\,, \tag{5.4}$$

where

$$\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T) := KCB\,\underline{u}_T - 2\,\mu\,\underline{p}_{k-1}\,, \tag{5.5}$$

$$\phi_{k-1}(\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T)) := \|\underline{\operatorname{dev}\tilde{\sigma}}_k(B\,\underline{u}_T)\|_N - \sigma_y(1 + H\alpha_{k-1})\,. \tag{5.6}$$

Since $D\,J_k(\underline{u}\,;\,\underline{v})$ is linear in $\underline{v}$, there exists the Fréchet-derivative

$$D\,J_k(\underline{u}) = \sum_{T\in\mathcal{T}} \left(|T| \left(CB\,\underline{u}_T - 2\mu\,\underline{\tilde{p}}_k(B\,\underline{u}_T) \right)^T B\,R_T - \underline{f}_T \right) - \sum_{E\in\mathcal{E}_N} \underline{g}_E\,. \tag{5.7}$$

Due to Corollary 4.13, the mapping $D\,J_k$ is slantly differentiable with

$$(D\,J_k)^o\,(\underline{u}) = \sum_{T\in\mathcal{T}} |T|\,R_T^T\,B^T \left(C - 2\mu\,\underline{\tilde{p}}_k^o(B\,\underline{u}_T) \right)^T B\,R_T\ ,$$

where

$$\underline{\tilde{p}}_k^o(B\,\underline{u}_T) = \begin{cases} \xi\,\left((1-\beta_k)\,\frac{\underline{\operatorname{dev}\tilde{\sigma}_k}\,\underline{\operatorname{dev}\tilde{\sigma}_k}^T N}{\|\underline{\operatorname{dev}\tilde{\sigma}_k}\|_N^2} + \beta_k I \right) KC & \text{if } \phi_k(\underline{\tilde{\sigma}_k}) > 0\,, \\ 0 & \text{else}\,. \end{cases} \tag{5.8}$$

serves as a slanting function for $\underline{\tilde{p}}_k$ defined in (5.4). Here, we use $\xi := \left(2\mu + \sigma_y^2 H^2\right)^{-1}$, $\beta_k := \phi_{k-1}(\underline{\operatorname{dev}\tilde{\sigma}_k})/\|\underline{\operatorname{dev}\tilde{\sigma}_k}\|_N$, and the short form $\underline{\operatorname{dev}\tilde{\sigma}_k}$ for $\underline{\operatorname{dev}\tilde{\sigma}_k}(B\,\underline{u}_T)$ as in (5.5).

5.2. The Slant Newton Method in Vector Representation. The slant Newton method is applied to calculate $\underline{u} \in \mathbb{R}^{2|\mathcal{N}|}$ such, that $DJ_k(\underline{u}) = 0$ and $\underline{u}$ satisfies the Dirichlet boundary condition. The iterates read

$$\underline{u}_j = \underline{u}_{j-1} + \Delta\underline{u}_j \qquad \forall\, j \in \mathbb{N}, \tag{5.9}$$

where $\Delta\underline{u}_j$ solves

$$(D\,J_k)^o\,(\underline{u}_{j-1})\;\Delta\underline{u}_j = -D\,J_k(\underline{u}_{j-1})\;.$$

Note, that $\underline{u}_j$ must satisfy (generally inhomogeneous) Dirichlet boundary conditions for all $j \in \mathbb{N}$. Therefore, it is sufficient for the initial approximation $\underline{u}_0$ to satisfy the inhomogeneous Dirichlet conditions, and for $\Delta\underline{u}_j$ to solve the homogeneous Dirichlet conditions. For the termination of the slant Newton method we check, if

$$\frac{|u_{h,j} - u_{h,j-1}|_\varepsilon}{|u_{h,j}|_\varepsilon + |u_{h,j-1}|_\varepsilon} \tag{5.10}$$

with $|\cdot|_\varepsilon := (\int_\Omega \|\varepsilon(\cdot)\|_F^2\;\mathrm{d}x)^{1/2}$ is smaller than a given prescribed bound $\epsilon > 0$. In the following section, this termination bound is set $\epsilon = 10^{-12}$.

6. Numerical Example. The following test example was calculated on a computer with 1.33 GHz CPU, 1024 KB cache size, 1 GB RAM using Matlab 7.0. It is taken from [28] and serves as a benchmark problem in computational plasticity. The example domain is a thin plate represented by the square $(-10, 10) \times (-10, 10)$ with a circular hole of the radius $r = 1$ in the middle, as can be seen in Figure 6.1. A surface load g is applied to the plate's upper and lower edge into normal direction. Just a single time step is considerd, thus the surface load with the intensity $|g| = 450$ is acting at once. Due to the symmetry of the domain, the solution has to be calculated on one quarter of the domain only. Therefore it is necessary to incorporate homogeneous Dirichlet boundary conditions in the normal direction (gliding conditions) to both symmetry axes. The material parameters are set

$$E = 206900\,,\quad \nu = 0.29\,,\quad \sigma_Y = 450\,\sqrt{2/3}\,,\quad H = 0.5\,.$$

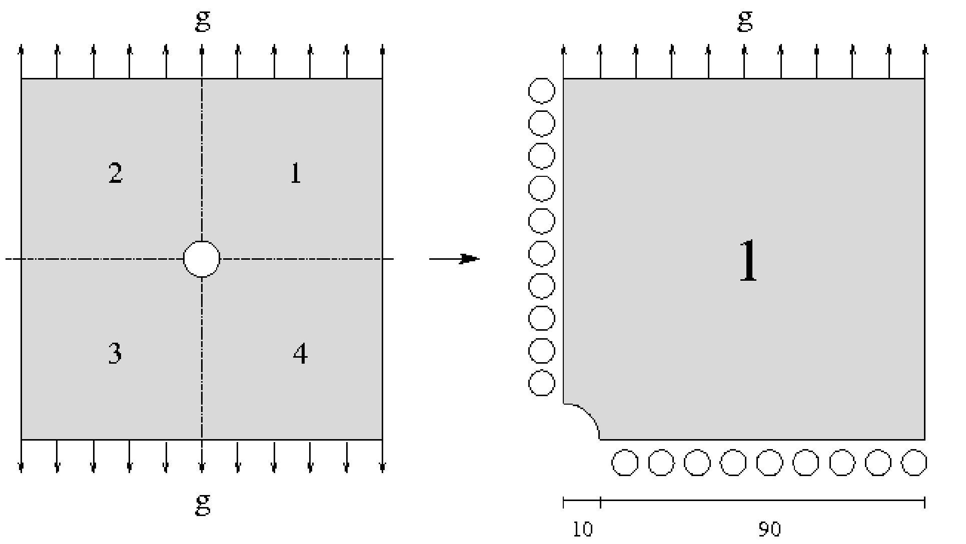

FIG. 6.1. *Here, the geometry of the example domain is outlined. Due to reasons of symmetry, the solution has to be calculated for one of the quarters only.*

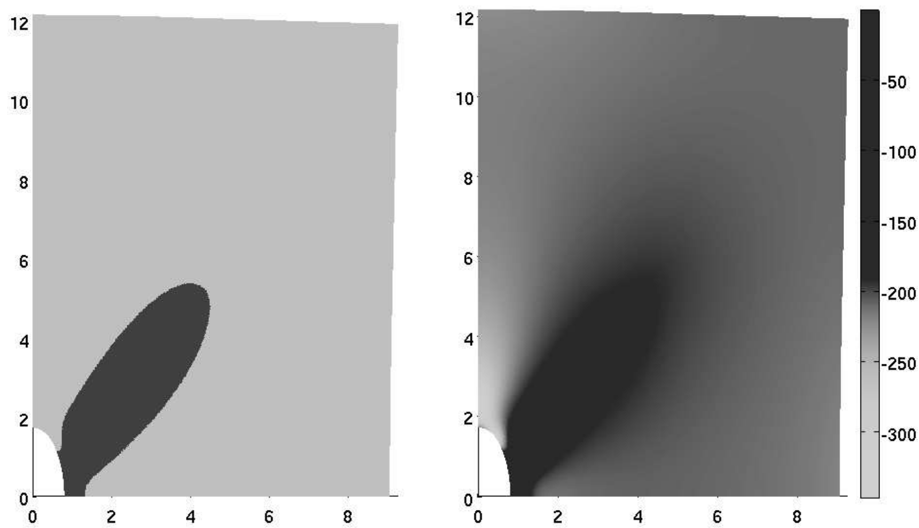

FIG. 6.2. *The two plots show the elastoplastic zones (left), and the yield function ϕ as in* (2.9) *(right) of the deformed domain. For better visibility, the displacement is magnified by a factor* 100.

Differently to the original problem in [28], we choose the modulus of hardening H to be nonzero, i. e. hardening effects are considered (cf. Remark 4.10). The numerical results concerning the application of the slant Newton method to the the original problem can be seen in [13, 12]. Figure 6.2 shows the yield function (right) and the elastic-plastic zones (left), where purely elastic zones are colored light gray, and plastic zones are colored dark gray. For better visibility, the displacement is multiplied by a factor 100. Table 6.1 reports on the convergence of the slant Newton method.

dof:	245	940	3680	14560	57920	231040
cri:						
0-1	1.669e-02	3.757e-02	2.564e-02	1.536e-02	8.133e-03	4.124e-03
1-2	3.456e-03	4.981e-03	2.559e-03	9.493e-04	2.478e-04	9.472e-05
2-3	2.650e-04	1.646e-04	5.274e-05	3.859e-05	8.815e-06	2.776e-06
3-4	4.976e-08	4.850e-08	1.814e-08	1.250e-07	1.147e-09	1.357e-08
4-5	5.581e-15	1.160e-14	5.741e-15	1.337e-13	3.872e-15	1.388e-14
res:						
0	1.637e+03	8.999e+02	7.471e+02	5.410e+02	3.423e+02	1.977e+02
1	4.182e+01	8.613e+01	2.619e+01	1.186e+01	3.144e+00	1.237e+00
2	3.463e+00	2.142e+00	5.991e-01	3.939e-01	1.207e-01	3.793e-02
3	6.948e-04	6.124e-04	2.209e-04	1.711e-03	1.234e-05	1.862e-04
4	6.878e-11	1.467e-10	7.575e-11	1.756e-09	1.687e-10	3.653e-10
5	1.290e-11	2.081e-11	4.131e-11	8.289e-11	1.698e-10	3.437e-10
sec:	2.85249	5.31937	14.6505	53.5494	226.626	1080.64

TABLE 6.1

This table outlines the convergence behavior of the slant Newton method. In horizontal direction, the refinement of the starting mesh takes place, where the degrees of freedom (`dof`) are approximately growing by a factor 4*. In the last line (`sec`) the computational time is displayed in seconds. Between the first and the last line, two blocks report on the quality of the slant Newton method. The first block (`cri`) shows the values* (5.10)*, corresponding to two subsequent Newton iteration steps $j-1$ and j. The second block (`res`) displays the norm of the residual, i. e. $\|D\,J(\underline{u}_j)\|$, in the jth Newton step. Both blocks show roughly super-linear convergence. Notice, that no more improvement of the residual takes place in the last iteration step. This is due to the fact, that the machine's accuracy has been reached. In this table, the slant Newton method takes its initial values from the interpolation of the solution on the previous level of refinement. This so called nested iteration strategy keeps the iteration steps constant during the different levels of refinement. Let be mentioned, that the number of iteration steps keeps roughly constant also if the initial values are chosen to be zero at all levels of refinement. The interested reader is referred to [13, 12, 14] for more convergence tables and numerical examples.*

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