Numerical Methods in Continuum Mechanics II

Tutorial 5

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14. Let Ω be a bounded domain and $u \in C^1(\Omega)$. Show, that the derivative $\mathrm{D}u$ of u serves as a weak derivative, and that for each arbitrary weak derivative u' of u there holds
$$\int_\Omega \big(u' - \mathrm{D}u\big)\, \varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\,.$$
That is, u' coincides with $\mathrm{D}u$ almost everywhere on Ω in the Lebesgue sense.

15. We define a function $u :]-1,1[\to \mathbb{R}$ by
$$u(x) = \begin{cases} u^-(x)\,, & \text{if } x \in]-1,0[\ , \\ \lim_{y\to 0} u^-(y)\,, & \text{if } x = 0\,, \\ u^+(x)\,, & \text{if } x \in]0,1[\ , \end{cases}$$
with $u^- \in C^1(]-1,0[)$ and $u^+ \in C^1(]0,1[)$ such that $\lim_{x\to 0} u^-(x) = \lim_{x\to 0} u^+(x)$. Is u weakly differentiable? If yes, then how many times is u weakly differentiable?

16. Consider the classical formulation of the Poisson problem: Find u such that for given f there holds
$$-\Delta u = f \text{ on } \Omega\,, \qquad\qquad u = 0 \text{ on } \partial\Omega\,,$$
where $\Delta u := \sum_i \frac{\partial^2 u}{\partial x_i^2}$. What are the minimal space requirements for u and f, such that this problem is well defined?

    Consider the related variational formulation: Find u such that
$$\int_\Omega \nabla u^T \nabla v \,\mathrm{d}x = \int_\Omega f v \mathrm{d}x \qquad \text{for all } v\,.$$
Again, what minimal space requirements for u, v and f do we have, such that this problem is well defined? Derive the variational formulation from the classical formulation and also do the other way round.

17. Differently as in Example 16 the boundary $\partial\Omega$ is split into two parts Γ_D and Γ_N (named after *Dirichlet* and *Neumann*). Now consider again the classical formulation of the Poisson problem: Find u such that for given f and g there holds
$$-\Delta u = f \text{ on } \Omega\,, \qquad u = 0 \text{ on } \Gamma_D\,, \qquad \partial_n u = g \text{ on } \Gamma_N\,,$$
where $\partial_n u := \nabla u^T n$ denotes the normal derivative. What does the related variational formulation look like?

Short survey on weak derivatives and Sobolev spaces Let Ω be a bounded domain. $C_0^\infty(\Omega)$ denotes the set of all C^∞ functions with compact support on Ω, i. e., there exists an open neighborhood of $\partial\Omega$ on which these functions are identically 0. Similarly, one can define such function space also for a subset $\Gamma \subset \partial\Omega$ instead of the whole boundary $\partial\Omega$: The function space $C_\Gamma^\infty(\Omega)$ is the set of all functions which vanish in an open neighborhood around Γ.

Definition 1. A function $u \in L_p(\Omega)$ is said to be *weakly differentiable in* $L_p(\Omega)$ if there exists a function $u' \in L_p(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ the identity

$$\int_\Omega u' \, \varphi \, \mathrm{d}x = - \int_\Omega u \, \mathrm{D}\varphi \, \mathrm{d}x$$

holds true. Such function u' is then called a *weak derivative* of u.

Definition 2. A function $u \in L_p(\Omega)$ is said to be k *times weakly differentiable in* $L_p(\Omega)$ if for every multi-indicex α with $|\alpha| \leq k$ there exists a function $u^{(\alpha)} \in L_p(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ the identity

$$\int_\Omega u^{(\alpha)} \, \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u \, \mathrm{D}_\alpha \varphi \, \mathrm{d}x$$

holds true.

Definition 2 motivates the definition of Sobolev spaces and norms

$$\begin{aligned}
W^{p,k}(\Omega) &:= \{v \in L_p(\Omega) \mid v \text{ is } k \text{ times weakly differentiable in } L_p(\Omega)\}\,, \\
W_0^{p,k}(\Omega) &:= \{v \in W^{p,k}(\Omega) \mid \exists\,(v_n) \in C_0^\infty(\Omega) : \lim_{n\to\infty} \|v - v_n\|_{W^{p,k}(\Omega)} = 0\}\,, \\
W_\Gamma^{p,k}(\Omega) &:= \{v \in W^{p,k}(\Omega) \mid \exists\,(v_n) \in C_\Gamma^\infty(\Omega) : \lim_{n\to\infty} \|v - v_n\|_{W^{p,k}(\Omega)} = 0\}\,,
\end{aligned}$$

$$\|v\|_{W^{p,k}(\Omega)} := \left(\sum_{|\alpha|\leq k} \|v^{(\alpha)}\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} , \qquad |v|_{W^{p,k}(\Omega)} := \left(\sum_{|\alpha|=k} \|v^{(\alpha)}\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} ,$$

$$H^k(\Omega) := W^{2,k}(\Omega)\,, \qquad H_0^k(\Omega) := W_0^{2,k}(\Omega)\,, \qquad H_\Gamma^k(\Omega) := W_\Gamma^{2,k}(\Omega)$$

$$\|v\|_k := \|v\|_{W^{2,k}(\Omega)}\,, \qquad |v|_k := |v|_{W^{2,k}(\Omega)}\,.$$

Notice, that there holds $W^{p,0}(\Omega) = W_0^{p,0}(\Omega) = L_p(\Omega)$, and that $H^k(\Omega)$, $H_0^k(\Omega)$ and $H_\Gamma^k(\Omega)$ are Hilbert spaces. Moreover, the quantity $|\cdot|_k$ is not a norm but a semi norm, which means that the implication $(|v|_k = 0) \Rightarrow (v = 0)$ is not valid.