Numerical Methods in Continuum Mechanics II

Tutorial 6

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Definition 1. On a vector space V, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent if there exist positive constants $\underline{c}$, $\overline{c} \in \mathbb{R}$ such that for all $v \in V$ there holds

$$\underline{c}\,\|v\|_1 \leq \|v\|_2 \leq \overline{c}\,\|v\|_1\,.$$

Definition 2. Let V be a vector space. A mapping $f : V \to [0, +\infty[$ is said to be a semi norm if it satisfies

1. $f(\lambda\, v) = |\lambda|\, f(v)\,, \quad \forall \lambda \in \mathbb{R}\,,\ \forall v \in V,$
2. $f(v + w) \leq f(v) + f(w)\,, \quad \forall v,\, w \in V.$

Theorem 1 (Sobolev's Theorem of Norm Equivalences)**.** *Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ be a bounded domain, let $l \in \mathbb{N}$, $k \in \mathbb{N}$ and $p \in [1, +\infty[$ be fixed arbitrarily, let for all $i \in \{1, 2, \ldots, l\}$ the mapping $f_i : W^{p,k}(\Omega) \to [0, +\infty[$ be a semi norm for which there exists a positive constant c_i such that $f_i(v) \leq c_i\, \|v\|_{W^{p,k}(\Omega)}$ holds for all $v \in W^{p,k}(\Omega)$, and let $\mathcal{P}_{k-1}(\mathbb{R}^n)$ denote the set of all polynomials of at most $k-1$st order defined on $\mathbb{R}^n$.*

If for all $v \in \mathcal{P}_{k-1}(\mathbb{R}^n)$ and $i \in \{1, 2, \ldots, l\}$ the implication $[f_i(v) = 0] \Rightarrow [v \equiv 0]$ is satisfied, then the norms $\|\cdot\|_{W^{p,k}(\Omega)}$ and $\|\cdot\|_{((f_i),p,k,\Omega)} := \left(\sum_{i=1}^{l} f_i(\cdot)^p + |\cdot|^p_{W^{p,k}(\Omega)}\right)^{1/p}$ are equivalent.

Examples:

18. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $\Gamma \subset \partial\Omega$ be of non-negative measure. Use Sobolev's Theorem of Norm Equivalences to show
    (a) Friedrichs' inequality: $\exists C_F \in\,]0, +\infty[\ \forall v \in H^1_0(\Omega):\ \|v\|_0 \leq C_F |v|_1,$
    (b) Friedrichs type inequality: $\exists C \in\,]0, +\infty[\ \forall v \in H^1_\Gamma(\Omega):\ \|v\|_0 \leq C |v|_1,$
    (c) Poincarè's inequality: $\exists C_P \in\,]0, +\infty[\ \forall v \in H^1(\Omega):\ \|v - v_\Omega\|_0 \leq C_P |v|_1,$

    where $v_\Omega := \frac{1}{|\Omega|} \int_\Omega v\, \mathrm{d}x$.

19. Consider the problems of Example 16 and 17 with $f \in L_2(\Omega)$ and $g \in L_2(\Gamma_N)$. Show, that each of the two variational formulations is uniquely solveable in $H^1_0(\Omega)$ and $H^1_{\Gamma_D}(\Omega)$, respectively.

20. Consider the classical formulation of the problem: Find u such that for given f and g there holds

    $$-\Delta u = f \text{ on } \Omega\,, \qquad\qquad \partial_n u = g \text{ on } \partial\Omega\,.$$

    Derive the related variational formulation, and show again, that for $f \in L_2(\Omega)$ and $g \in L_2(\partial\Omega)$, the variational formulation has a unique solution u amongst functions $v \in H^1(\Omega)$ which satisfy $\int_\Omega v\, \mathrm{d}x = 0$.