Numerical Methods in Continuum Mechanics II

Tutorial 7

December 6, 2007

Galerkin scheme Let V be a infinite dimensional vectorspace, e.g., $H^1(\Omega)$, and let $n \in \mathbb{N}$ be fixed arbitarily. The problem is to find $u \in V$ such that $a(u, v) = \langle F\,,\, v\rangle$ holds for all $v \in V$. To use the Galerkin scheme means to solve the problem not in V but in an n-dimensional subspace V_h. We identify V_h by a basis $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ and solve: Find $u_h := \sum_{i=1}^n u_i\varphi_i \in V_h$ such that for all $v_h := \sum_{i=1}^n v_i\varphi_i \in V_h$ with u_i and v_i in $\mathbb{R}$ for all i there holds $a(u_h, v_h) = \langle F\,,\, v_h\rangle$. Such problem can be rewritten as: Find $\underline{u}_h \in \mathbb{R}^n$ such that for all $\underline{v}_h \in \mathbb{R}^n$ there holds

$$\underline{v}_h^T K_h \underline{u}_h = \underline{v}_h^T \underline{f}_h \tag{1}$$

with

$$\underline{u}_h := (u_i)_{i=1}^n\ , \quad \underline{v}_h := (v_i)_{i=1}^n\ , \quad K_h := (a(\varphi_j, \varphi_i))_{i,j=1}^n\ , \quad \underline{f}_h := (\langle F\,,\, \varphi_i\rangle)_{i=1}^n\ .$$

Since (1) must hold for all $\underline{v}_h \in \mathbb{R}^n$ it suffices to solve $K_h\underline{u}_h = \underline{f}_h$.

21. Generate the Finite Element system for the mixed boundary value problem

$$\begin{aligned}
-\Delta u(x_1, x_2) &= 0 \quad \forall\, (x_1, x_2) \in \Omega := \,]0, 1[\, \times \,]0, 1[\ , && (2)\\
u(x_1, 0) &= 0 \quad \forall\, x_1 \in [0, 1]\ , && (3)\\
u(0, x_2) &= 0 \quad \forall\, x_2 \in [0, 1]\ , && (4)\\
\frac{\partial}{\partial x_1} u(1, x_2) &= x_2 \quad \forall\, x_2 \in \,]0, 1]\ , && (5)\\
\frac{\partial}{\partial x_2} u(x_1, 1) &= x_1 \quad \forall\, x_1 \in \,]0, 1[\ , && (6)
\end{aligned}$$

under use of the triangulation in Figure 1 under use of linear triangle elements, i. e. $V_h = \{\varphi_1(x_1, x_2), \varphi_2(x_1, x_2), \ldots, \varphi_9(x_1, x_2)\}$ such that $\varphi_i(x_1, x_2)$ is linear on each triangle and there holds $\varphi_i(\mathbf{v}_j) = \delta_{ij}$ on the node $\mathbf{v}_j$ for all i and j. Solve the linear FE-system.

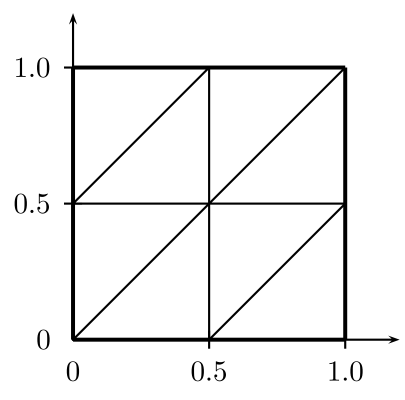

Figure 1: $\Omega = \,]0, 1[\, \times \,]0, 1[$.

22. Solve the variational problem: Find $u \in V := L_2(0,1)$ such that

$$\int_0^1 u(x)v(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x \qquad \forall\, v \in V\,,$$

with the Galerkin scheme by using the monomial basis

$$V_h = \mathrm{span}\{1, x, x^2, x^3, \ldots, x^{n-1}\}\,.$$

Calculate the stiffness matrix K_h analytically and solve the linear system $K_h \underline{u}_h = \underline{f}_h$ numerically by using the Gaussian algorithm in Matlab (in order to solve the system $A\,x = b$, one would use the commands: `[L,U]=lu(A); x = U\(L\b);`). Therefore, let $f(x) := \cos((k+1)\pi x)$ with k denoting the last digit of your students id. Vary $n \in \{2, 4, 8, 10, 50, 100\}$.

23. Show that the integration formula

$$\int_\Delta f(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \approx \frac{1}{2} f(x_1^*, x_2^*)$$

with $\Delta = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 1 - x_1\,,\ 0 < x_1 < 1\}$ and $x_1^* = x_2^* = 1/3$ is exact if and only if f is in $\mathcal{P}_1(\Delta)$, i. e., a linear polynomial.

24. Show that the integration formula

$$\int_\Delta f(\xi, \eta)\mathrm{d}\xi\mathrm{d}\eta \approx \frac{1}{2}\left(\alpha_1 f(\xi_1, \eta_1) + \alpha_2 f(\xi_2, \eta_2) + \alpha_3 f(\xi_3, \eta_3)\right)$$

with $\Delta = \{(\xi, \eta) \in \mathbb{R}^2 \mid 0 < \eta < 1 - \xi\,,\ 0 < \xi < 1\}$ and

(a) $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, $(\xi_1, \eta_1) = (1/2, 0)$, $(\xi_2, \eta_2) = (1/2, 1/2)$, $(\xi_3, \eta_3) = (0, 1/2)$,

(b) $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, $(\xi_1, \eta_1) = (1/6, 1/6)$, $(\xi_2, \eta_2) = (4/6, 1/6)$, $(\xi_3, \eta_3) = (1/6, 4/6)$,

is exact if and only if f is in $\mathcal{P}_2(\Delta)$, i. e., a quadratic polynomial.