29. Let $V$ denote a function space and $V_D \subset V$ its subset (hyper plane) of functions which satisfy the Dirichlet condition $v = u_D$ on $\Gamma_D$. Let $a(\cdot, \cdot)$ be a bilinear form, $\langle F, \cdot \rangle$ a linear form, $\psi : V_D \to \mathbb{R}$ a convex function, and $J(v) := \frac{1}{2} a(v, v) + \psi(v) - \langle F, v \rangle$.

Show that $u \in V_D$ solves the variational inequality

$$a(u, u - v) + \psi(u) - \psi(v) \leq \langle F, u - v \rangle \quad \forall v \in V_D,$$

if and only if it satisfies

$$J(u) = \inf_{v \in V_D} J(v).$$

30. When modeling elastoplasticity with the kinematic hardening scheme (cf. lectures) the bilinearform of the variational inequality reads

$$a((u, p), (v, q)) = \int_{\Omega} C (\varepsilon(u) - p) : (\varepsilon(v) - q) + H p : q \, dx.$$

Here, $C$ is the elastic stiffness tensor (Hooke’s law), and $H$ is a positive definite tensor of the same size and dimension as $C$. Show that there exists $\alpha > 0$ such that $a((v, q), (v, q)) \geq \alpha \left( \|v\|_{H^1}^2 + \|q\|_{L^2}^2 \right)$ holds for all $(v, q) \in [H^1_0(\Omega)]^3 \times [L^2(\Omega)]^{3 \times 3}_{\text{sym}}$.

31. Let $A \in \mathbb{R}^{3 \times 3}$ with $\text{tr} A = 0$ and $b \in \mathbb{R}$ with $b > 0$. Find $Q$ with $\text{tr} Q = 0$ such that

$$J(Q) := \frac{1}{2} Q : Q - A : Q + b \|Q\|_F$$

attains its minimum.