On Schwarz-type Smoothers for Saddle Point Problems with Applications to PDE-Constrained Optimization Problems

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Abstract

In this paper we consider a (one-shot) multigrid strategy for solving the discretized optimality system (KKT system) of a PDE-constrained optimization problem. In particular, we discuss the construction of an additive Schwarz-type smoother for a certain class of optimal control problems. A rigorous multigrid convergence analysis is presented. Numerical experiments are shown which confirm the theoretical results.

1 Introduction

In this paper we discuss multigrid methods for solving large-scale systems of discretized mixed variational problems. The main applications considered here are optimization problems in function spaces with constraints in form of partial differential equations (PDEs). The necessary (and for the problems considered here also sufficient) first-order optimality conditions on a solution of such a problem can be written as a mixed variational problem, usually called the optimality system or Karush-Kuhn-Tucker (KKT) system.

In particular, we will consider elliptic optimal control problems, see, e.g., [13], [17]. In such problems the primal unknown, say x, consists of two parts: a function y, the so-called state, and a function u, the so-called control. The problem is to find $x = (y, u)$ from appropriate function spaces that minimizes a given cost functional subject to a constraint, the so-called state equation, which, for each control u, is an elliptic boundary value problem in y. The corresponding KKT system involves another (dual) unknown, say p (the Lagrangian multiplier or the adjoint state), and consists of three components: the state equation (see above), the adjoint state equation, which, for each state y, is an elliptic boundary value

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problem in p, and the control equation, which is typically an algebraic relation between u and p.

In principle, there are two different approaches for mixed problems, such as KKT systems, to take advantage of the multigrid idea. One way is to use an outer iteration, typically a preconditioned Richardson method (possibly accelerated by a Krylov subspace method), applied to the discretized mixed problem. For typical preconditioners of KKT systems in elliptic optimal control, see, e.g., [2], [3], [4], [12] and the references cited there. These preconditioners usually rely on efficient solvers or preconditioners for the state equation (as a PDE in y) and the adjoint state equation (as a PDE in p) and on the construction of a good preconditioner for the corresponding Schur complement of the KKT system, which is the reduced Hessian of the Lagrangian. A preconditioner based on a different Schur complement is proposed in [15]. Multigrid techniques (as an inner iteration or approximation) can be used for (some or all of) these components, see, e.g., [10], [9], [15].

The other way is to use multigrid methods directly applied to the (discretized) mixed problem as an outer iteration based on appropriate smoothers (as a sort of inner iteration). For PDE-constrained optimization problems this approach is also known as one-shot multigrid strategy, see [16]. One of the most important ingredients of such a multigrid method is an appropriate smoother.

A first approach for constructing such smoothers is to combine standard smoothers applied to the components elliptic state and adjoint equations complemented with a special relaxation method for the control equation, see, e.g., [1].

A second class of smoothers are point smoothers, where the variable, here y, u and p, are grouped pointwise (with respect to the points (nodes) of the underlying mesh) and one or several sweeps of point-block Jacobi or point-block Gauß-Seidel sweeps with respect to this grouping are performed, see, e.g, [5].

A natural extension of point smoothers are patch smoothers: The computational domain is divided into small (overlapping or non-overlapping) patches. One iteration step of the smoothing process consists of solving local mixed problems on each patch one-by-one either in a Jacobi-type or Gauss-Seidel-type manner. This results in an additive or multiplicative Schwarz-type smoother. The technique was successfully used for the Navier-Stokes equations, see [18]. The general construction and the analysis of patch smoothers for mixed problems was discussed in [14], where a particular patch smoother was proposed for the Stokes problem. An essential feature exploited in the multigrid convergence analysis of the Stokes problem was (in the terminology introduced here) that the adjoint state equation is an elliptic problem in y, where in elliptic optimal control problems the adjoint state equation is typically elliptic in p but not necessarily in y. Therefore, a straight forward application of this construction to KKT systems for elliptic optimal control problems fails.

Another well-known class of smoothers for mixed problems are Braess-Sarazin smoothers, see, [6], [21], which are well suited for Stokes-like problems but are too expensive if applied to KKT systems of elliptic optimal control problems.

A last approach discussed here for constructing smoothers for mixed problems leads to so-called transforming smoothers, see [19], [20], which were successfully analyzed for the Stokes problem and the Navier-Stokes problem. They still lack a rigorous analysis for more general mixed problems like the KKT systems of elliptic optimal control problems.

So far, the multigrid convergence analysis for KKT systems of PDE-constrained optimization problems is not as developed as for elliptic PDEs. One line of argument exploits the fact that the KKT system reduced to y and p by eliminating u using the control equation is a compact perturbation of an elliptic problem. This guarantees the convergence of the multigrid method if the coarse grid is sufficiently fine, see [5]. A second strategy is based on a Fourier analysis, which, strictly speaking, covers only the case of uniform meshes with special boundary conditions (and small perturbations of this situation), see, e.g., [5], [1].

The aim of this paper is to contribute to the one-shot multigrid approach for KKT systems. We will modify the construction of the patch smoother discussed in [14] for KKT systems of elliptic optimal control problems and present a rigorous convergence analysis of the corresponding multigrid method.

In order to keep the notations simple and the strategy transparent the material is presented for a model problem in optimal control only. However, since the construction of the method does not rely on structural information of some Schur complement, the method is easily applicable to more general problems.

The paper is organized as follows: In Section 2 the class of problems, the multigrid method and the Schwarz-type smoother are introduced. Section 3 contains the roadmap to prove the multigrid convergence in an abstract setting. The main part of the paper is the application to PDE-constrained optimization problems in Section 4, the proof of the approximation property in Section 5, and the proof of the smoothing property in Section 6, which completes the multigrid convergence analysis. Finally, in Section 7 some numerical results are presented, followed by some concluding remarks on the extension to more general mixed problems.

2 The framework

Here we follow mainly the notations introduced in [7] and [14] and give a short review of the general results on Schwarz-type smoothers from [14]:

Let X and Q be real Hilbert spaces, $a\colon X \times X \longrightarrow \mathbb{R}$, $b\colon X \times Q \longrightarrow \mathbb{R}$, $c\colon Q \times Q \longrightarrow \mathbb{R}$ continuous bilinear forms, and $F\colon X \longrightarrow \mathbb{R}$, $G\colon Q \longrightarrow \mathbb{R}$ continuous linear functionals.

We consider the following mixed variational problem: Find $x \in X$ and $p \in Q$ such that

$$\begin{aligned} a(x,w) + b(w,p) &= \langle F, w\rangle \quad \text{for all } w \in X,\\ b(x,q) - c(p,q) &= \langle G, q\rangle \quad \text{for all } q \in Q. \end{aligned}$$

Here, $\langle F, w\rangle$ ($\langle G, q\rangle$) denotes the evaluation of the linear functional F (G) at the point w (q).

The mixed variational problem can also be written as a variational problem on $X \times Q$: Find $(x,p) \in X \times Q$ such that

$$\mathcal{B}((x,p),(w,q)) = \langle \mathcal{F},(w,q)\rangle \quad \text{for all } (w,q) \in X \times Q \tag{1}$$

with the bilinear form

$$\mathcal{B}((x,p),(w,q)) = a(x,w) + b(w,p) + b(x,q) - c(p,q)$$

and the linear functional

$$\langle \mathcal{F}(w,q)\rangle = \langle F,w\rangle + \langle G,q\rangle.$$

It is assumed that a and c are symmetric and non-negative and that $\mathcal{B}$ is stable on $X \times Q$.

Typical examples of this type of problems are the Stokes problem, various problems from linear elasticity, mixed formulations of boundary value problems for second order elliptic equations, see e.g. Brezzi, Fortin [8], and, in particular, PDE-constrained optimization problems, see Section 4.

The Hilbert spaces X and Q are typically subspaces of Sobolev spaces on some domain Ω. Then, for discretizing the continuous problem (1), a sequence of finite element spaces X_k and Q_k, symmetric bilinear forms $\mathcal{B}_k$ and linear functionals $\mathcal{F}_k$ on $X_k \times Q_k$ are chosen for each level $k = 1, 2, \ldots$, corresponding to a hierarchy of increasingly finer meshes on Ω.

These spaces, linear and bilinear forms determine discrete problems at each level k: Find $(x_k, p_k) \in X_k \times Q_k$ such that

$$\mathcal{B}_k((x_k,p_k),(w,q)) = \langle \mathcal{F}_k,(w,q)\rangle \quad \text{for all } (w,q) \in X_k \times Q_k. \tag{2}$$

A class of efficient solvers of these discrete problems are multigrid algorithms: We additionally need coarse-to-fine inter-grid transfer operators $I_{k-1}^k \colon X_{k-1} \times Q_{k-1} \longrightarrow X_k \times Q_k$. Then one iteration step for solving (2) at level k is given in the following form:

Let $(x_k^{(0)}, p_k^{(0)}) \in X_k \times Q_k$ be a given approximation of the exact solution $(x_k, p_k) \in X_k \times Q_k$ to (2). Then the iteration proceeds in two stages:

1. Smoothing: For $j = 0, 1, \ldots, m-1$ compute $(x_k^{(j+1)}, p_k^{(j+1)}) \in V_k \times Q_k$ by an iterative procedure of the form
$$(x_k^{(j+1)}, p_k^{(j+1)}) = \mathcal{S}_k\,(x_k^{(j)}, p_k^{(j)}).$$

2. Coarse grid correction: Set
$$\langle \tilde{\mathcal{F}}_{k-1},(w,q)\rangle = \langle \mathcal{F}_k, I_{k-1}^k(w,q)\rangle - \mathcal{B}_k\left((x_k^{(m)}, p_k^{(m)}), I_{k-1}^k(w,q)\right)$$
for $(w,q) \in X_{k-1} \times Q_{k-1}$ and let $(\tilde{s}_{k-1}, \tilde{r}_{k-1}) \in V_{k-1} \times Q_{k-1}$ satisfy
$$\mathcal{B}_{k-1}((\tilde{s}_{k-1}, \tilde{r}_{k-1}),(v,q)) = \langle \tilde{\mathcal{F}}_{k-1},(w,q)\rangle \quad \text{for all } (w,q) \in X_{k-1} \times Q_{k-1}. \tag{3}$$
If $k = 1$, compute the exact solution of (3) and set $(s_{k-1}, r_{k-1}) = (\tilde{s}_{k-1}, \tilde{r}_{k-1})$.

If $k > 1$, compute approximations (s_{k-1}, r_{k-1}) by applying $\mu \geq 2$ iteration steps of the multigrid algorithm applied to (3) on level $k-1$ with zero starting values.

Set

$$(x_k^{(m+1)}, p_k^{(m+1)}) = (x_k^{(m)}, p_k^{(m)}) + I_{k-1}^k (s_{k-1}, r_{k-1}).$$

Next we will describe the smoothing procedure in detail. For this it will be more convenient to use matrix-vector notation: Let $w \in X_k$ and $q \in Q_k$. Then $\underline{w} \in \mathbb{R}^{n_k}$ and $\underline{q} \in \mathbb{R}^{m_k}$ denote their vector representations (i.e. the vectors of coefficients relative to some bases in X_k and Q_k). Furthermore, we introduce the matrix representation of the bilinear forms by

$$\mathcal{B}_k((s,r),(w,q)) = (A_k\underline{s}, \underline{w})_{\ell^2} + (B_k\underline{w}, \underline{r})_{\ell^2} + (B_k\underline{s}, \underline{q})_{\ell^2} - (C_k\underline{r}, \underline{q})_{\ell^2},$$

and the vector representation of the linear forms

$$\langle \mathcal{F}_k, (w,q) \rangle = (\underline{f}_k, \underline{w})_{\ell^2} + (\underline{g}_k, \underline{q})_{\ell^2}.$$

Here $(.,.)_{\ell^2}$ denotes the Euclidean scalar product, whose associated vector norm and matrix norm will both be denoted by $\|.\|_{\ell^2}$.

In matrix-vector notation the discrete problem (2) can be written as:

$$\mathcal{K}_k \begin{pmatrix} \underline{x}_k \\ \underline{p}_k \end{pmatrix} = \begin{pmatrix} \underline{f}_k \\ \underline{g}_k \end{pmatrix} \quad \text{with} \quad \mathcal{K}_k = \begin{pmatrix} A_k & B_k^T \\ B_k & -C_k \end{pmatrix}.$$

Here, B_k^T denotes the transpose of the matrix B_k. We assume that A_k and C_k are symmetric positive semi-definite matrices, and that $\mathcal{K}_k$ is a nonsingular matrix.

Since the smoothing procedure involves only one level k of the hierarchy of spaces, we will simplify the notation for the rest of this section by dropping the subscript k and, additionally, omitting underlining the vectors. So, from now on, we discuss iterative methods (as smoothers) for linear systems of equations of the form:

$$\mathcal{K} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{with} \quad \mathcal{K} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \tag{4}$$

where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, under the assumption that A is a symmetric positive semi-definite $n \times n$ matrix, B is an $m \times n$ matrix, and C is a symmetric positive semi-definite $m \times m$ matrix, and that $\mathcal{K}$ is nonsingular.

For setting up local sub-problems a set of linear operators is introduced:

$$P_i \colon \mathbb{R}^{n_i} \longrightarrow \mathbb{R}^n, \quad Q_i \colon \mathbb{R}^{m_i} \longrightarrow \mathbb{R}^m, \quad \text{for } i = 1, \ldots, N,$$

where the dimensions n_i and m_i are typically much smaller than the dimensions n and m of the original spaces, respectively. The operators P_i and Q_i are interpreted as prolongation operators with associated restriction operators P_i^T and Q_i^T. We assume that

$$\sum_{i=1}^N Q_i Q_i^T \text{ is nonsingular} \quad \text{and} \quad \sum_{i=1}^N P_i P_i^T = I, \tag{5}$$

where I denotes the identity matrix. These conditions guarantee that we have complete space decompositions

$$\mathbb{R}^n = \sum_{i=1}^N P_i(\mathbb{R}^{n_i}) \quad \text{and} \quad \mathbb{R}^m = \sum_{i=1}^N Q_i(\mathbb{R}^{m_i}),$$

and that, additionally, the prolongations P_i determine a special partition of unity: For each $u \in \mathbb{R}^n$ we have

$$u = \sum_{i=1}^N P_i u_i \quad \text{with} \quad u_i = P_i^T u \quad \text{and} \quad \|u\|_{\ell^2}^2 = \sum_{i=1}^N \|P_i u_i\|_{\ell^2}^2.$$

For each index $i \in 1, \ldots, N$, local matrices $\hat{A}_i$, B_i and $\hat{S}_i$ of size $n_i \times n_i$, $m_i \times n_i$ and $m_i \times m_i$, respectively, have to be chosen, which determine local matrices $\hat{\mathcal{K}}_i$ of the form

$$\hat{\mathcal{K}}_i = \begin{pmatrix} \hat{A}_i & B_i^T \\ B_i & B_i \hat{A}_i^{-1} B_i^T - \hat{S}_i \end{pmatrix}.$$

We assume that the local matrices B_i are related to the (global) matrix B by the following commutativity condition

$$Q_i^T B = B_i P_i^T \quad \text{for all } i = 1, 2, \ldots, N. \tag{6}$$

A similar condition is needed for the local matrices $\hat{A}_i$: We assume that there exists a (global) matrix $\hat{A}$ (typically not equal to A) such that:

$$P_i^T \hat{A} = \hat{A}_i P_i^T \quad \text{for all } i = 1, 2, \ldots, N. \tag{7}$$

From the local matrices $\hat{S}_i$ the following (global) matrix $\hat{S}$ is constructed:

$$\hat{S} = \left(\sum_{i=1}^N Q_i \hat{S}_i^{-1} Q_i^T \right)^{-1}. \tag{8}$$

With the help of the local saddle point matrices $\hat{\mathcal{K}}_i$ the following iterative method is constructed: Starting from some approximations $x^{(j)}$ and $p^{(j)}$ of the exact solutions x and p of (4) we consider iterative methods of form:

$$x^{(j+1)} = x^{(j)} + \sum_{i=1}^N P_i\, s_i^{(j)}, \quad p^{(j+1)} = p^{(j)} + \sum_{i=1}^N Q_i\, r_i^{(j)},$$

where $(s_i^{(j)}, r_i^{(j)}) \in \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}$ solves a local saddle point problem of the form

$$\hat{\mathcal{K}}_i \begin{pmatrix} s_i^{(j)} \\ r_i^{(j)} \end{pmatrix} = \begin{pmatrix} P_i^T \left[f - A x^{(j)} - B^T p^{(j)} \right] \\ Q_i^T \left[g - B x^{(j)} + C p^{(j)} \right] \end{pmatrix} \quad \text{for all } i = 1, \ldots, N.$$

That means, that the residuals of the approximations are first restricted to the smaller spaces, then a series of small saddle point problems must be solved, and, finally, the solutions are prolongated and determine the next iterate. This process can be viewed as an additive Schwarz method.

It was shown in [14] that, under the assumptions (5), (6) and (7) and with the construction (8), this iterative method can be written equivalently as the following preconditioned Richardson method:

$$x^{(j+1)} = x^{(j)} + s^{(j)}, \quad p^{(j+1)} = p^{(j)} + r^{(j)}, \tag{9}$$

where $s^{(j)}$, $r^{(j)}$ solve the equation

$$\hat{\mathcal{K}} \begin{pmatrix} s^{(j)} \\ r^{(j)} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{K} \begin{pmatrix} x^{(j)} \\ p^{(j)} \end{pmatrix} \quad \text{with} \quad \hat{\mathcal{K}} = \begin{pmatrix} \hat{A} & B^T \\ B & B\hat{A}^{-1}B^T - \hat{S} \end{pmatrix}. \tag{10}$$

So the additive Schwarz-type iterative method can be represented as an symmetric inexact Uzawa method. Let $\mathcal{M}$ denote the associated iteration matrix, given by

$$\mathcal{M} = I - \hat{\mathcal{K}}^{-1}\mathcal{K},$$

which controls the error propagation for the iterative method.

In the next theorem an important estimate is formulated which is needed in the forthcoming multigrid convergence analysis. Here and in the sequel the following notations are used: $M < N$ ($N > M$) iff $N - M$ is positive definite, and $M \leq N$ ($N \geq M$) iff $N - M$ is positive semi-definite, for symmetric matrices M and N. Furthermore, for a symmetric and positive definite matrix S, the norms $\|v\|_S$ and $\|M\|_S$ of a vector v and a matrix M (as a representation of a bilinear form) are given by

$$\|v\|_S = \sqrt{(Sv, v)_{\ell^2}} \quad \text{and} \quad \|M\|_S = \sup_{v,w\neq 0} \frac{|(Mv, w)_{\ell^2}|}{\|v\|_S \, \|w\|_S}.$$

Theorem 1. *Let A be a symmetric and positive semi-definite $n \times n$ matrix, B an $m \times n$ matrix, and C be a symmetric and positive semi-definite $m \times m$ matrix. Let $\hat{A}$ be a symmetric and positive definite $n \times n$ matrix, and $\hat{S}$ a symmetric positive definite $m \times m$ matrix, satisfying*

$$\hat{A} \geq A \quad \textit{and} \quad \hat{S} \geq C + B\hat{A}^{-1}B^T.$$

Then

$$\|\mathcal{K}\mathcal{M}^m\|_{\mathcal{L}} \leq \eta_0(m) \, \|\mathcal{Q}\|_{\mathcal{L}},$$

where $\mathcal{K}$ is given by (4), $\hat{\mathcal{K}}$ is given by (10), $\mathcal{Q}$ is given by

$$\mathcal{Q} = \begin{pmatrix} \hat{A} - A & 0 \\ 0 & \hat{S} - C - B\hat{A}^{-1}B^T \end{pmatrix},$$

$\mathcal{L}$ is an arbitrary symmetric and positive definite matrix, and

$$\eta_0(m) = \frac{1}{2^{m-1}} \binom{m-1}{[m]/2} \leq \begin{cases} \sqrt{\dfrac{2}{\pi(m-1)}} & \textit{for even } m, \\ \sqrt{\dfrac{2}{\pi m}} & \textit{for odd } m. \end{cases}$$

Here $\binom{n}{k}$ denotes the binomial coefficient and $[x]$ denotes the largest integer smaller than or equal to $x \in \mathbb{R}$.

Proof. For the special case of the Euclidean norm ($\mathcal{L} = I$) see [14], the proof for more general norms is completely analogous. $\square$

3 Multigrid convergence analysis

A classical technique for analyzing the convergence of multigrid methods relies on two properties: the approximation property and the smoothing property, see Hackbusch [11], which will be discussed in this section.

First we need an appropriate (L^2-like) mesh-dependent norm $|||(w,q)|||_{0,k}$ on $X_k \times Q_k$. We introduce a second discrete norm on $X_k \times Q_k$ by

$$|||(s,r)|||_{2,k} = \sup_{0 \neq (w,q) \in X_k \times Q_k} \frac{|\mathcal{B}_k((s,r),(w,q))|}{|||(w,q)|||_{0,k}}.$$

Now, we can formulate the approximation property and the smoothing property: Consider the two-grid algorithm (i.e. exact solution of the coarse grid correction equation (3) at level $k-1$). The approximation property measures the effect of the coarse grid correction: It is assumed that there is a constant c_A which is independent of k such that

$$|||\mathcal{B}_k|||_{0,k}\, |||(x_k^{(m+1)} - x_k, p_k^{(m+1)} - p_k)|||_{0,k} \leq c_A\, |||(x_k^{(m)} - x_k, p_k^{(m)} - p_k)|||_{2,k}, \qquad (11)$$

where the norm $|||\mathcal{B}_k|||_{0,k}$ of the bilinear form $\mathcal{B}_k$ is given by

$$|||\mathcal{B}_k|||_{0,k} = \sup_{0 \neq (s,r),(w,q) \in X_k \times Q_k} \frac{|\mathcal{B}_k((s,r),(w,q))|}{|||(s,r)|||_{0,k}\, |||(w,q)|||_{0,k}}.$$

The remaining part to complete the proof of the two-grid convergence is the smoothing property, which measures the effect of the smoothing procedure: It is assumed that

$$|||(x_k^{(m)} - x_k, p_k^{(m)} - p_k)|||_{2,k} \leq \eta(m)\, |||\mathcal{B}_k|||_{0,k}\, |||(x_k^{(0)} - x_k, p_k^{(0)} - p_k)|||_{0,k}$$

for some function $\eta(m)$ which is independent of k, and

$$\eta(m) \to 0 \quad \text{for } m \to \infty.$$

The convergence of the two-grid method for a sufficiently large number m of smoothing steps easily follows by combining the approximation property and the smoothing property. From this the convergence of the multigrid method can be derived by standard arguments, see, e.g., Hackbusch [11].

Let $\mathcal{L}_k$ be the symmetric and positive definite matrix on $\mathbb{R}^{n_k} \times \mathbb{R}^{m_k}$ which represents the mesh dependent norm $|||(w,q)|||_{0,k}$:

$$|||(w,q)|||_{0,k} = \left(\left(\mathcal{L}_k \begin{pmatrix} \underline{w} \\ \underline{q} \end{pmatrix}, \begin{pmatrix} \underline{w} \\ \underline{q} \end{pmatrix} \right)_{\ell^2} \right)^{1/2} = \left\| \begin{pmatrix} \underline{w} \\ \underline{q} \end{pmatrix} \right\|_{\mathcal{L}_k} \qquad (12)$$

for $w \in X_k$, $q \in Q_k$ with vector representations $\underline{w} \in \mathbb{R}^{n_k}$, $\underline{q} \in \mathbb{R}^{m_k}$.

It easily follows that the smoothing property translates to the following condition in matrix-notation:

$$\|\mathcal{K}_k \mathcal{M}_k^m\|_{\mathcal{L}_k} \leq \eta(m)\, \|\mathcal{K}_k\|_{\mathcal{L}_k}. \tag{13}$$

Comparing with Theorem 1 it is immediately clear that the smoothing property (13) is satisfied for the additive Schwarz-type method introduced in the previous section, if the local problems are constructed in such a way that the associated global matrices $\hat{A}_k$, B_k and $\hat{S}_k$, see (7), (6) and (8), satisfy the conditions

$$\hat{A}_k \geq A_k \quad \text{and} \quad \hat{S}_k \geq C_k + B_k \hat{A}_k^{-1} B_k^T \tag{14}$$

and if, additionally, the following scaling condition holds:

$$\|\mathcal{Q}_k\|_{\mathcal{L}_k} \leq c_R\, \|\mathcal{K}_k\|_{\mathcal{L}_k} \quad \text{with} \quad \mathcal{Q}_k = \begin{pmatrix} \hat{A}_k - A_k & 0 \\ 0 & \hat{S}_k - C_k - B_k \hat{A}_k^{-1} B_k^T \end{pmatrix} \tag{15}$$

for some constant c_R independent of k. The smoothing rate is then given by $\eta(m) = c_R\, \eta_0(m) = O(1/\sqrt{m})$.

In [14] this strategy was successfully applied to the Stokes problem, discretized by the Taylor-Hood mixed finite element method. For the global matrix $\hat{A}_k$ at level k, required in Condition (7), a constant multiple of $\operatorname{diag} A_k$ was chosen. Special local matrices were constructed and all requirements of the analysis could be verified. In particular, the scaling condition (15) could be shown. In the next section the application to a typical class of problems from optimal control is discussed. For this class, the same choice for $\hat{A}_k$ as a constant multiple of $\operatorname{diag} A_k$ leads to a violation of the scaling condition (15). It will be shown how the construction must be modified to keep the right scaling without loosing any of the other requirements.

4 Application to an optimal control problem

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$. Let $L^2(\Omega)$ and $H^1(\Omega)$ denote the usual Lebesgue space and Sobolev space, respectively. We consider the following elliptic optimal control problem: Find the state $y \in H^1(\Omega)$ and the control $u \in L^2(\Omega)$ such that

$$J(y, u) = \min_{(z,v) \in H^1(\Omega) \times L^2(\Omega)} J(z, v)$$

with cost functional

$$J(z, v) = \frac{1}{2}\|z - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|v\|^2_{L^2(\Omega)}$$

subject to the state equations

$$\begin{aligned} -\Delta y + y &= u && \text{in } \Omega, \\ \frac{\partial y}{\partial n} &= 0 && \text{on } \Gamma, \end{aligned}$$

where Γ denotes the boundary of Ω, $y_d \in L^2(\Omega)$ is the desired state and $\nu > 0$ is the weight of the cost of the control (or simply a regularization parameter).

By introducing the adjoint state $p \in H^1(\Omega)$ we get the following optimality system, see e.g., [17]:

1. The adjoint state equation:

$$\begin{aligned} -\Delta p + p &= -(y - y_d) \quad \text{in } \Omega, \\ \frac{\partial p}{\partial n} &= 0 \qquad\qquad\quad \text{on } \Gamma. \end{aligned} \tag{16}$$

2. The control equation:

$$\nu u - p = 0 \quad \text{in } \Omega. \tag{17}$$

3. The state equation:

$$\begin{aligned} -\Delta y + y &= u \quad \text{in } \Omega, \\ \frac{\partial y}{\partial n} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{18}$$

The weak formulation of this problem leads to a mixed variational problem: Find $x = (y, u) \in X = Y \times U$ with $Y = H^1(\Omega)$, $U = L^2(\Omega)$ and $p \in Q = H^1(\Omega)$ such that

$$\begin{aligned} a(x, w) + b(w, p) &= \langle F, w \rangle \qquad \text{for all } w \in X, \\ b(x, q) &= 0 \qquad\qquad\; \text{for all } q \in Q \end{aligned} \tag{19}$$

with

$$\begin{aligned} a(x, w) &= (y, z)_{L^2(\Omega)} + \nu(u, v)_{L^2(\Omega)}, \\ b(w, q) &= (z, q)_{H^1(\Omega)} - (v, q)_{L^2(\Omega)}, \\ \langle F, w \rangle &= (y_d, z)_{L^2(\Omega)}, \end{aligned}$$

where $w = (z, v)$ with $z \in Y$, $v \in U$, and $(., .)_H$ is the standard scalar product in a Hilbert space H, whose norm is denoted by $\|.\|_H$.

The mixed variational problem can also be written as a variational problem on $X \times Q$: Find $(x, p) \in X \times Q$ such that

$$\mathcal{B}((x, p), (w, q)) = \langle \mathcal{F}, (w, q) \rangle \quad \text{for all } (w, q) \in X \times Q$$

with the bilinear form

$$\mathcal{B}((x, p), (w, q)) = a(x, w) + b(w, p) + b(x, q)$$

and the linear functional

$$\langle \mathcal{F}(w, q) \rangle = \langle F, w \rangle.$$

It is trivial that $\mathcal{F}$ is a bounded and linear functional on $X \times Q$. The next lemma guarantees that the problem is well-posed:

Lemma 1. *The bilinear form $\mathcal{B}$ is stable and bounded on $X \times Q$, i.e., there are positive constants c and C such that*

$$c\,\|(s,r)\|_{X\times Q} \le \sup_{0\neq(w,q)\in X\times Q} \frac{\mathcal{B}((s,r),(w,q))}{\|(w,q)\|_{X\times Q}} \le C\,\|(s,r)\|_{X\times Q} \quad \textit{for all } (s,r)\in X\times Q,$$

where the norm on $X \times Q$ is given by

$$\|(w,q)\|_{X\times Q}^2 = \|w\|_X^2 + \|q\|_Q^2$$

with

$$\|w\|_X^2 = \|z\|_{H^1(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \quad \textit{for } w=(z,v) \quad \textit{and} \quad \|q\|_Q = \|q\|_{H^1(\Omega)}.$$

Proof. Boundedness follows easily from Cauchy's inequality, the stability from Brezzi's Theorem, see [8], if the bilinear form a is coercive on the set

$$\ker B = \{w \in X : b(w,q) = 0 \text{ for all } q \in Q\},$$

i.e.: there exists a constant $\alpha > 0$ such that

$$a(w,w) \ge \alpha\,\|w\|_X^2 \quad \text{for all } w \in \ker B, \tag{20}$$

and the inf-sup condition for b is satisfied, i.e.: there exists a constant $\beta > 0$ such that

$$\sup_{0\neq w\in X} \frac{b(w,q)}{\|w\|_X} \ge \beta\,\|q\|_Q \quad \text{for all } q \in Q. \tag{21}$$

To show (20) let $w = (z,v) \in \ker B$. Then, in particular, we have $b(w,z) = 0$, i.e.

$$(z,z)_{H^1(\Omega)} = (v,z)_{L^2(\Omega)},$$

which implies

$$\|z\|_{H^1(\Omega)}^2 \le \|v\|_{L^2(\Omega)}\,\|z\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}\,\|z\|_{H^1(\Omega)}.$$

Hence: $\|z\|_{H^1(\Omega)} \le \|v\|_{L^2(\Omega)}$. Then

$$a(w,w) = \|z\|_{L^2(\Omega)}^2 + \nu\,\|v\|_{L^2(\Omega)}^2 \ge \|z\|_{L^2(\Omega)}^2 + \nu\,\|z\|_{H^1(\Omega)}^2 \ge \nu\,\|z\|_{H^1(\Omega)}^2,$$

which proves (20) with $\alpha = \nu$.

Next we have

$$\sup_{0\neq w\in X} \frac{b(w,q)}{\|w\|_X} \ge \frac{b((q,0),q)}{\|(q,0)\|_X} = \frac{(q,q)_Q}{\|q\|_Q} = \|q\|_Q,$$

which shows (21) with $\beta = 1$. □

Let $(\mathcal{T}_k)$ be a sequence of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by dividing each triangle into four smaller triangles by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. The quantity $\max\{\operatorname{diam} T : T \in \mathcal{T}_k\}$ is denoted by h_k.

We consider the following discretization by continuous and piecewise linear finite elements:

$$\begin{aligned} X_k = Y_k \times U_k &= \{(z,v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : z|_T, v|_T \in P_1 \text{ for all } T \in \mathcal{T}_k\}, \\ Q_k &= \{q \in C(\bar{\Omega}) : q|_T \in P_1 \text{ for all } T \in \mathcal{T}_k\}, \end{aligned}$$

where P_1 denotes the polynomials of total degree less or equal to 1. Then we obtain the following discrete variational problem: Find $x_k \in X_k$ and $p_k \in Q_k$ such that

$$\begin{aligned} a(x_k, w_k) + b(w_k, p_k) &= \langle F, w_k \rangle \qquad \text{for all } w_k \in X_k, \\ b(x_k, q_k) &= 0 \qquad\qquad\quad \text{for all } q_k \in Q_k. \end{aligned} \tag{22}$$

The discrete mixed variational problem can also be written as a discrete variational problem on $X_k \times Q_k$: Find $(x_k, p_k) \in X_k \times Q_k$ such that

$$\mathcal{B}((x_k, p_k), (w_k, q_k)) = \langle \mathcal{F}, (w_k, q_k) \rangle \quad \text{for all } (w_k, q_k) \in X_k \times Q_k. \tag{23}$$

With exactly the same arguments as above the boundedness and stability of the discrete mixed problem can be shown with constants c and C independent of k:

Lemma 2. *The bilinear form $\mathcal{B}$ is uniformly stable and bounded on $X_k \times Q_k$, i.e., there are positive constants c and C such that*

$$c\,\|(s,r)\|_{X\times Q} \le \sup_{0\ne(w,q)\in X_k\times Q_k} \frac{\mathcal{B}((s,r),(w,q))}{\|(w,q)\|_{X\times Q}} \le C\,\|(s,r)\|_{X\times Q} \quad \textit{for all } (s,r) \in X_k \times Q_k,$$

for all k.

By introducing the standard nodal basis, we finally obtain the following saddle point problem in matrix-vector notation:

$$\mathcal{K}_k \begin{pmatrix} \underline{x}_k \\ \underline{p}_k \end{pmatrix} = \begin{pmatrix} \underline{f}_k \\ 0 \end{pmatrix} \quad \text{with} \quad \mathcal{K}_k = \begin{pmatrix} A_k & B_k^T \\ B_k & 0 \end{pmatrix}$$

where

$$A_k = \begin{pmatrix} M_k & 0 \\ 0 & \nu\, M_k \end{pmatrix} \quad \text{and} \quad B_k = \begin{pmatrix} K_k & -M_k \end{pmatrix},$$

Here M_k denotes the mass matrix representing the $L^2(\Omega)$ scalar product on Y_k, and K_k denotes the stiffness matrix representing the $H^1(\Omega)$ scalar product on Y_k.

5 Approximation property

Here we follow the general technique presented in [7] for proving the approximation property (11) for the special mesh dependent norm on $X_k \times Q_k$, given by

$$|||(w, q)|||_{0,k} = h_k \left(\|\underline{z}\|_{\ell^2} + h_k^2 \|\underline{v}\|_{\ell^2} + \|\underline{q}\|_{\ell^2} \right)^{1/2} \tag{24}$$

with $w = (z, v) \in X_k$ and $q \in Q_k$ and their vector representations $\underline{z}$, $\underline{v}$ and $\underline{q}$.

Nine assumptions (A.1) - (A.9) are formulated in [7] which imply the approximation property. Assumptions (A.1), (A.2), (A.4) and (A.5) are the (uniform) boundedness and stability of the continuous and discrete variational problem which were already shown in Lemma 1 and Lemma 2.

The verification of Assumption (A.3) is the content of the next lemma:

Lemma 3. *For all $f, g \in L^2(\Omega)$ the variational problem*

$$\mathcal{B}((x, p), (w, q)) = (f, z)_{L^2(\Omega)} + (g, q)_{L^2(\Omega)} \quad \text{for all } w = (z, v) \in X = Y \times U,\ q \in Q \tag{25}$$

has a solution $x = (y, u) \in H^2(\Omega) \times H^1(\Omega)$ and $q \in H^2(\Omega)$. There exists a constant C such that

$$\|y\|_{H^2(\Omega)} + \|u\|_{H^1(\Omega)} + \|p\|_{H^2(\Omega)} \le C\,(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}).$$

for all $f, g \in L^2(\Omega)$.

Proof. From (25) with $v = 0$ and $q = 0$ we obtain the elliptic variational problem for p:

$$(p, z)_{H^1(\Omega)} = (f, z)_{L^2(\Omega)} + (y, z)_{L^2(\Omega)} \quad \text{for all } z \in H^1(\Omega).$$

Then elliptic regularity and Lemma 1 imply

$$\|p\|_{H^2(\Omega)} \le C\,(\|f\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}) \le C\,(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}).$$

(Througout this proof, C denotes a generic constant.) From (25) with $z = 0$ and $v = 0$ we obtain the elliptic variational problem for y:

$$(y, q)_{H^1(\Omega)} = (g, q)_{L^2(\Omega)} + (u, q)_{L^2(\Omega)} \quad \text{for all } q \in H^1(\Omega).$$

Then elliptic regularity and Lemma 1 imply

$$\|y\|_{H^2(\Omega)} \le C\,(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \le C\,(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}).$$

From (25) with $z = 0$ and $q = 0$ we obtain

$$\nu\, u = p.$$

Then Lemma 1 implies

$$\|u\|_{H^1(\Omega)} = \frac{1}{\nu}\,\|p\|_{H^1(\Omega)} \le C\,(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}).$$

□

Assumption (A.6) requires the standard L^2 estimate of the approximation error of the finite element space, which is, of course, satisfied.

Assumption (A.7) requires an L^2 discretization error estimate for problem (25), which easily follows from (A.1) - (A.6) by standard arguments (Aubin-Nitsche duality trick), since the introduced finite element discretization is a conforming method.

Assumption (A.8) requires the equivalence of the mesh-dependent norm with an L^2 norm:

$$|||(w,q)|||_{0,k} \sim \left(\|z\|^2_{L^2(\Omega)} + h_k^2 \|v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right)^{1/2} \quad \text{with } w = (z,v).$$

(The symbol $\sim$ denotes the equivalence of norms.) This is easy to see by standard scaling arguments.

Finally Assumption (A.9) on the inter-grid transfer operators is trivial since the subspaces are nested.

So, in summary, we have:

Theorem 2. *The approximation property (11) is satisfied for the mesh-dependent norm, given by (24).*

6 Smoothing property

For the smoothing procedure we have to define appropriate local sub-problems at grid level k. So the prolongation operators and the matrices of the local sub-problems must be specified for the optimal control problem.

Let N_k be the number of nodes of the triangulation $\mathcal{T}_k$. Then we have $n_k = 2N_k$ degrees of freedom for the primal variable $x_k = (y_k, u_k) \in X_k$ and $m_k = N_k$ degrees of freedom for the dual variable $p_k \in Q_k$.

We will now define a space decomposition of $\mathbb{R}^{n_k} \times \mathbb{R}^{m_k}$ into N_k subspaces: For each $i \in \{1, \ldots, N_k\}$ representing a node of the triangulation, let $\mathcal{N}_{k,i}$ be the set of all indices consisting of i and the indices of all neighboring nodes (all nodes which are connected to the node with index i by an edge of the triangulation). Then, for each $i \in \{1, \ldots, N_k\}$, the associated local patch consists of all unknowns of y_k and u_k which are associated to nodes with indices from $\mathcal{N}_{k,i}$ and the unknown of p_k which is associated to the node with index i, see Figure 1 for an illustration of a local patch. The corresponding prolongations are the canonical embeddings into $\mathbb{R}^{n_k}$ and $\mathbb{R}^{m_k}$ and are denoted by $\hat{P}_{k,i}$ and $Q_{k,i}$, respectively.

Observe that all entries in $\hat{P}_{k,i}$ and $Q_{k,i}$ are either 0 or 1. A single component of p_k belongs to exactly one patch, while a single component of y_k or u_k belongs, in general, to more than one patch. It is easy to see that

$$\sum_{i=1}^{N_k} Q_{k,i} Q_{k,i}^T = I \quad \text{and} \quad \sum_{i=1}^{N_k} \hat{P}_{k,i} \hat{P}_{k,i}^T = \mathcal{D}_k,$$

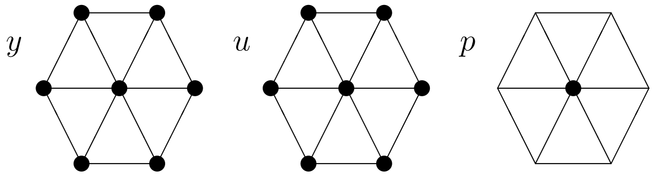

Figure 1: local patches

where

$$\mathcal{D}_k = \begin{pmatrix} D_k & 0 \\ 0 & D_k \end{pmatrix}$$

with the $N_k \times N_k$ diagonal matrix D_k whose diagonal entries $d_{k,j}$ are the local overlap depth at the node with index j, i.e., the number of all indices l with $j \in \mathcal{N}_{k,l}$, for $j = 1, \ldots, N_k$. Observe that

$$1 \le d_{k,j} \le d_{\max} \quad \text{for all } j = 1, \ldots, N_k \tag{26}$$

with a constant $d_{\max}$ independent of the grid level k.

In order to guarantee Condition (5), we have to scale the prolongations $\hat{P}_{k,i}$ accordingly:

$$P_{k,i} = \mathcal{D}_k^{-1/2}\, \hat{P}_{k,i}.$$

Next we have to choose a matrix $\hat{A}_k$, needed on the left-hand side in Condition 7. It seems to be natural to choose

$$\hat{A}_k = \frac{1}{\sigma} \operatorname{diag} A_k = \frac{1}{\sigma} \begin{pmatrix} \operatorname{diag} M_k & 0 \\ 0 & \operatorname{diag} M_k \end{pmatrix} \tag{27}$$

with a suitable parameter $\sigma > 0$ as it was done in [14] for the Stokes problem. But in order to prove the smoothing property, we have to check, if the estimate (15) is fulfilled. This is not the case with this definition of $\hat{A}_k$ for parameters $\sigma = O(1)$. Instead we choose:

$$\hat{A}_k = \frac{1}{\sigma} \begin{pmatrix} \operatorname{diag} K_k & 0 \\ 0 & \operatorname{diag} M_k \end{pmatrix} \tag{28}$$

with σ small enough to ensure

$$\hat{A}_k \ge A_k.$$

Since $K_k \ge M_k$ and the maximal number of non-zero entries per row in M_k and K_k is bounded by a constant, say nnz_x, independently of k, it suffices to choose $\sigma = 1/nnz_x$. Here we use the estimate $M \le nnz(M) \operatorname{diag} M$ for all symmetric and positive definite matrices M, where $nnz(M)$ denotes the maximum number of non-zero entries per row in M.

For the local sub-problems we choose just the restriction of $\hat{A}_k$ to those components of $\underline{x}_k$ whose indices are in $\mathcal{N}_{k,i}$:

$$\hat{A}_{k,i} = \hat{P}_{k,i}^T \hat{A}_k \hat{P}_{k,i}. \tag{29}$$

Since the matrices $\hat{A}_k$ and $\hat{A}_{k,i}$ are diagonal the condition (7) is satisfied.

The other matrices of the local sub-problems are specified similarly: For

$$B_{k,i} = Q_{k,i}^T B_k \mathcal{D}_k^{1/2}\, \hat{P}_{k,i} \tag{30}$$

one can verify the relation (6), since

$$B_{k,i} P_{k,i}^T = Q_{k,i}^T B_k \mathcal{D}_k^{1/2}\, \hat{P}_{k,i} \hat{P}_{k,i}^T \mathcal{D}_k^{-1/2}$$

and the i-th component of $B_k v$ depends only on the degrees of freedom on the surrounding triangles, whose indices are collected in the set $\mathcal{N}_{k,i}$. On this index set $\hat{P}_{k,i}\hat{P}_{k,i}^T$ acts like the identity.

Finally, we set

$$\hat{S}_{k,i} = \frac{1}{\tau}\, B_{k,i} \hat{A}_{k,i}^{-1} B_{k,i}^T$$

with τ small enough to guarantee

$$\hat{S}_k \geq B_k \hat{A}_k^{-1} B_k^T. \tag{31}$$

Using (29), (30), and (26), we get

$$\hat{S}_{k,i} = \frac{1}{\tau}\, Q_{k,i}^T B_k \mathcal{D}_k^{1/2} \hat{A}_k^{-1} \mathcal{D}_k^{1/2} B_k^T Q_{k,i} \geq \frac{1}{\tau}\, Q_{k,i}^T B_k \hat{A}_k^{-1} B_k^T Q_{k,i}$$

and with (8)

$$\hat{S}_k \geq \frac{1}{\tau}\, \mathrm{diag}(B_k \hat{A}_k^{-1} B_k^T). \tag{32}$$

So, in order to guarantee (31) it suffices to choose τ such that

$$\frac{1}{\tau}\, \mathrm{diag}(B_k \hat{A}_k^{-1} B_k^T) \geq B_k \hat{A}_k^{-1} B_k^T.$$

Since the maximal number of non-zero entries per row in $B_k \hat{A}_k^{-1} B_k^T$ is bounded by a constant, say nnz_p, independent of k it suffices to choose $\tau = 1/nnz_p$.

The matrix $\mathcal{L}_k$ representing the mesh-dependent norm $|||.|||_{0,k}$ is given by

$$\mathcal{L}_k = h_k^2 \begin{pmatrix} I & 0 & 0 \\ 0 & h_k^2 I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

The last missing part for the smoothing property (13) is the estimate (15), which will be shown in the next lemma:

Lemma 4. *With the setting from above there is a constant c_R, independent of k, such that*

$$\|\mathcal{Q}_k\|_{\mathcal{L}_k} \leq c_R\, \|\mathcal{K}_k\|_{\mathcal{L}_k} \quad \textit{with} \quad \mathcal{Q}_k = \begin{pmatrix} \hat{A}_k - A_k & 0 \\ 0 & \hat{S}_k - C_k - B_k \hat{A}_k^{-1} B_k^T \end{pmatrix}.$$

Proof. Since

$$0 \le \mathcal{Q}_k = \begin{pmatrix} \hat{A}_k - A_k & 0 \\ 0 & \hat{S}_k - B_k \hat{A}_k^{-1} B_k^T \end{pmatrix} \le \begin{pmatrix} \hat{A}_k & 0 \\ 0 & \hat{S}_k \end{pmatrix}$$

we have

$$\|\mathcal{K}_k\|_{\mathcal{L}_k} \le \left\| \begin{pmatrix} \hat{A}_k & 0 \\ 0 & \hat{S}_k \end{pmatrix} \right\|_{\mathcal{L}_k}.$$

Using the block diagonal form of $\hat{A}_k$ and $\mathcal{L}_k$ we obtain

$$\left\| \begin{pmatrix} \hat{A}_k & 0 \\ 0 & \hat{S}_k \end{pmatrix} \right\|_{\mathcal{L}_k} = \frac{1}{h_k^2} \max \left(\frac{1}{\sigma} \| \operatorname{diag} K_k \|_{\ell^2}, \frac{\nu}{\sigma\, h_k^2} \| \operatorname{diag} M_k \|_{\ell^2}, \|\hat{S}_k\|_{\ell^2} \right).$$

It follows from (26) that

$$\hat{S}_{k,i} = \frac{1}{\tau} Q_{k,i}^T B_k \mathcal{D}_k^{1/2} \hat{A}_k^{-1} \mathcal{D}_k^{1/2} B_k^T Q_{k,i} \le \frac{d_{\max}}{\tau} Q_{k,i}^T B_k \hat{A}_k^{-1} B_k^T Q_{k,i}$$

and, therefore,

$$\hat{S}_k \le \frac{d_{\max}}{\tau} \operatorname{diag}(B_k \hat{A}_k^{-1} B_k^T).$$

Using this estimate and the simple general estimates $\| \operatorname{diag} M\|_{\ell^2} \le \|M\|_{\ell^2}$ for any matrix M and $\|M\|_{\ell^2} \le \|N\|_{\ell^2}$ for symmetric matrices M, N with $0 \le M \le N$ we obtain

$$\left\| \begin{pmatrix} \hat{A}_k & 0 \\ 0 & \hat{S}_k \end{pmatrix} \right\|_{\mathcal{L}_k} \le \frac{1}{h_k^2} \max \left(\frac{1}{\sigma} \|K_k\|_{\ell^2}, \frac{\nu}{\sigma\, h_k^2} \|M_k\|_{\ell^2}, \frac{d_{max}}{\tau} \|B_k \hat{A}_k^{-1} B_k^T\|_{\ell^2} \right).$$

Furthermore,

$$\begin{aligned} \|B_k \hat{A}_k^{-1} B_k^T\|_{\ell^2} &= \|\sigma K_k (\operatorname{diag} K_k)^{-1} K_k + \frac{\sigma}{\nu} M_k (\operatorname{diag} M_k)^{-1} M_k\|_{\ell^2} \\ &\le \|\sigma K_k (\operatorname{diag} K_k)^{-1} K_k\|_{\ell^2} + \frac{\sigma}{\nu} \|M_k (\operatorname{diag} M_k)^{-1} M_k\|_{\ell^2} \\ &\le \sigma\, nnz_x \|K_k\|_{\ell^2} + \frac{\sigma\, nnz_x}{\nu} \|M_k\|_{\ell^2} \\ &= \|K_k\|_{\ell^2} + \frac{1}{\nu} \|M_k\|_{\ell^2} \le \left(1 + \frac{1}{\nu}\right) \|K_k\|_{\ell^2}. \end{aligned}$$

Summarizing the estimates from above we obtain

$$\|\mathcal{Q}_k\|_{\mathcal{L}_k} \le \frac{c_R}{h_k^2} \max \left(\|K_k\|_{\ell^2}, \frac{\nu}{h_k^2} \|M_k\|_{\ell^2} \right)$$

with $c_R = \max(1/\sigma, (1 + 1/\nu) d_{\max}/\tau)$. Finally, using the fact that the spectral norm of a block matrix is greater or equal to the spectral norm of each of its sub-blocks, it follows that

$$\|\mathcal{K}_k\|_{\mathcal{L}_k} \ge \frac{1}{h_k^2} \max \left(\|K_k\|_{\ell^2}, \frac{\nu}{h_k^2} \|M_k\|_{\ell^2} \right),$$

which completes the proof. $\square$

So, in summary, we have

Theorem 3. *For the additive Schwarz smoother constructed in this section the smoothing property (13) holds with a smoothing rate* $\eta(m) = O(1/\sqrt{m})$.

7 Numerical Experiments

Next we present some numerical results for the domain $\Omega = (0,1)\times(0,1)$ and homogeneous data $y_d = 0$. The initial grid consists of two triangles by connecting the nodes $(0,0)$ and $(1,1)$. For the first series of experiments the regularization parameter ν was set equal to 1. The dependence of the convergence rate on the regularization parameter ν was investigated subsequently.

Randomly chosen starting values for $x_k^{(0)}$ and $p_k^{(0)}$ for the exact solution $x_k = 0$ and $p_k = 0$ were used. The discretized problem was solved by a multigrid iteration with a W-cycle ($\mu = 2$) and $m/2$ pre- and $m/2$ post-smoothing steps. The multigrid iteration was performed until the Euclidean norm of the solution was reduced by a factor $\varepsilon = 10^{-8}$.

Table 1 contains the total number of unknowns n_k+m_k, the number of iterations *it* and the (average) convergence rates q depending on the level k and the number m of smoothing steps. It shows a typical multigrid convergence behavior, namely the independence of the grid level and the expected improvement of the rates with an increasing number of smoothing steps.

		smoothing steps							
level	$n_k + m_k$	5+5		7+7		10+10		15+15	
5	3 267	50	0.70	35	0.59	24	0.46	16	0.31
6	12 675	53	0.71	37	0.61	26	0.49	18	0.35
7	49 923	54	0.71	38	0.61	27	0.50	18	0.36
8	198 147	54	0.71	38	0.61	27	0.50	18	0.36
9	789 507	54	0.71	38	0.61	27	0.50	18	0.36

Table 1: Convergence rates for the additive Schwarz smoother

Table 2 shows the convergence rates with the multiplicative version of the smoother. As expected, the rates are significantly better than the rates for the additive smoother. The number of smoothing steps which are necessary to achieve convergence on all levels is much smaller than in the additive version.

For comparison, convergence rates are shown for the additive Schwarz smoother based on (27) instead of (28) in Table 3. There is a significant increase in the number of iterations and there is no clear indication of level-independent convergence rates. This underlines the significance of the modification in the construction of the smoother compared to the original construction in [14].

		smoothing steps							
level	$n_k + m_k$	5+5		7+7		10+10		15+15	
5	3 267	34	0.57	26	0.49	15	0.28	12	0.19
6	12 675	40	0.62	29	0.52	16	0.30	12	0.20
7	49 923	40	0.63	29	0.52	16	0.31	12	0.21
8	198 147	41	0.64	28	0.51	16	0.31	12	0.21
9	789 507	41	0.64	28	0.51	16	0.31	12	0.21

Table 2: Convergence rates for the multiplicative Schwarz smoother

		smoothing steps							
level	$n_k + m_k$	5+5		7+7		10+10		15+15	
5	3 267	84	0.80	60	0.73	41	0.63	29	0.52
6	12 675	105	0.83	75	0.77	52	0.70	35	0.58
7	49 923	119	0.85	84	0.80	58	0.72	38	0.60
8	198 147	133	0.86	91	0.81	62	0.73	41	0.63
9	789 507	139	0.87	96	0.82	66	0.75	43	0.65

Table 3: Convergence rates for the unmodified additive Schwarz smoother

All numerical experiments shown so far were performed for the regularization parameter $\nu = 1$. Observe that the analysis presented here does not predict convergence rates that are robust in ν. And indeed, numerical experiments indicate a mild dependence of the convergence rate on the regularization parameter ν, see Table 4. The results were obtained at grid level 9 with 10+10 smoothing steps.

ν	it	q
1	27	0.50
10^{-2}	33	0.57
10^{-4}	40	0.63

Table 4: Dependence on the regularization parameter ν

In summary, the numerical experiments confirm the theoretical results of a level-independent convergence rate for the multigrid method with the additive Schwarz smoother. The multiplicative smoother leads to better rates, however, a theoretical analysis for the convergence and smoothing properties is still missing. The modification of the local problems compared to previous work, see [14], leads to a significant improvement. The convergence rates depend only mildly on the regularization parameter.

8 Concluding remarks

The basic idea of constructing a Schwarz-type smoother carries over to a much wider class of mixed variational problems: An essential step in the construction of the local problems is the choice of the matrix $\hat{A}_k$. Instead of using for $\hat{A}_k$ a multiple of the diagonal part of the discretization matrix A_k corresponding to the bilinear form $a : X \times X \longrightarrow \mathbb{R}$ one should choose the diagonal part of the discretization matrix corresponding to a positive bilinear form on X which is coercive and bounded on the whole space X instead. For the Stokes problem the bilinear form a is already positive, coercive and bounded on X, so the diagonal of A_k will do the job. For the presented optimal control problem the bilinear form a is coercive only on $\operatorname{Ker} B$, therefore, $\hat{A}_k$ was chosen differently.

The local matrices are then constructed by restricting the global matrices to the local patches. The requirements of the analysis in terms of certain commutativity relations of the involved global and local matrices determine the size of the patches and the overlap. Additionally, an appropriate scaling is required which takes into account the local overlap depth of the components of the primal variables.

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