SUPERCONVERGENCE PROPERTIES FOR OPTIMAL CONTROL PROBLEMS DISCRETIZED BY PIECEWISE LINEAR AND DISCONTINUOUS FUNCTIONS

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Abstract. An optimal control problem for an elliptic equation is investigated with pointwise control constraints. This paper is concerned with discretization of the control by piecewise linear but discontinuous functions. The state and the adjoint state are discretized by linear finite elements. Approximations of the optimal solution of the continuous optimal control problem will be constructed by a projection of the discrete adjoint state. It is proved that these approximations have convergence order h^2.

Keywords: Linear-quadratic optimal control problems, error estimates, elliptic equations, numerical approximation, control constraints, superconvergence.

AMS subject classification: 49K20, 49M25, 65N30

1. Introduction. The paper is concerned with the discretization of the elliptic optimal control problem

$$J(u) = F(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to the state equations

$$\begin{aligned} Ay + a_0 y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma \end{aligned} \tag{1.2}$$

and subject to the control constraints

$$a \le u(x) \le b \qquad \text{for a.a. } x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain and Γ is the boundary of Ω; A denotes a second order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{N} D_i(a_{ij}(x) D_j y(x))$$

where D_i denotes the partial derivative with respect to x_i, and a and b are real numbers. Moreover, $\nu > 0$ is a fixed positive number. We denote the set of admissible controls by U_{ad}:

$$U_{ad} = \{u \in L^2(\Omega) : \ a \le u \le b \ \text{ a.e. in } \Omega\}.$$

We discuss here the full discretization of the control and the state equations by a finite element method. The asymptotic behavior of the discretized problem is studied, and superconvergence results are established.

The approximation of the discretization for semilinear elliptic optimal control problems is discussed in Arada, Casas, and Tröltzsch [1]. The optimal control problem (1.1)–(1.3) is a linear-quadratic

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counterpart of such a semilinear problem. Our aim is to construct controls $\tilde{u}$ which have an approximation order of h^2. This higher convergence order is the difference to [1].

The discretization of optimal control problems by piecewise constant functions is well investigated, we refer to Falk [7], Geveci [8]. Piecewise constant and piecewise linear discretization in space are discussed for a parabolic problem in Malanowski [12]. Theory and numerical results for elliptic boundary control problems are contained in Casas and Tröltzsch [5] and Casas, Mateos, and Tröltzsch [4].

Piecewise linear control discretizations for elliptic optimal control problems are studied by Casas and Tröltzsch, see [5]. In an abstract optimization problem, piecewise linear approximations are investigated in Rösch [15]. In all papers, the convergence order is h or $h^{3/2}$.

A quadratic convergence result is proved by Hinze [10]. In that approach only the state equation is discretized. The control is obtained by a projection of the adjoint state to the set of admissible controls.

In this paper, we combine the advantages of the different approaches. After solving a fully discretized optimal control problem, a control $\tilde{u}$ is calculated by the projection of the adjoint state p_h in a post-processing step. Although the approximation of the discretized solution is only of order $h^{3/2}$, we will show that this post-processing step improves the convergence order to h^2. This idea was already used in the paper [13] for piecewise constant control functions. The authors want to point out, that the main idea of the proof cannot be applied to other types of functions (especially piecewise linear controls). This concerns in particular the derivation of formulas (3.11), (3.13), and (4.2). These formulas are the main tool in the proof of the superconvergence results in that paper. Thus, a direct transfer of the ideas in [13] seems to be impossible. In contrast to [13] we will introduce an auxiliary control that is constructed using both the solution of the continuous and the discretized optimal control problem. Here we need an approximation result in the L^∞-norm, see [16].

The paper is organized as follows: In Section 2 the discretizations are introduced and the main results are stated. Section 3 contains auxiliary results. The proofs of the superconvergence results are given in Section 4. The paper ends with numerical experiments shown in Section 5.

2. Discretization and superconvergence results. Throughout this paper, Ω denotes a convex bounded open subset in $\mathbb{R}^N$ with $N = 2, 3$ of class $C^{1,1}$. The coefficients a_{ij} of the operator A belong to $C^{0,1}(\bar{\Omega})$ and satisfy the ellipticity condition

$$m_0|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \qquad \forall (\xi, x) \in \mathbb{R}^N \times \bar{\Omega}, \quad m_0 > 0.$$

Moreover, we require $a_{ij}(x) = a_{ji}(x)$ and $y_d \in L^p(\Omega)$ for some $p > N$. For the function $a_0 \in L^\infty(\Omega)$, we assume $a_0 \ge 0$. Next, we recall a result from Grisvard [9], Theorem 2.4.2.5.

LEMMA 2.1. *[9] For every $p > N$ and every function $g \in L^p(\Omega)$, the solution y of*

$$Ay + a_0 y = g \qquad \text{in } \Omega, \quad y|_\Gamma = 0,$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant c, independent of a_0 such that

$$\|y\|_{W^{2,p}(\Omega)} \le c\|g\|_{L^p(\Omega)}.$$

Consequently, a solution operator S of (1.2) is defined mapping the control u to the state y, i.e., $y = Su$. Although we have seen that the operator S maps from $L^p(\Omega)$ to $W^{2,p}(\Omega)$, we will investigate this operator in other function spaces in particular as operator acting in $L^2(\Omega)$.

Next, we introduce the adjoint equation

$$\begin{aligned} Ap + a_0 p = y - y_d \qquad & \text{in } \Omega \\ p = 0 \qquad & \text{on } \Gamma \end{aligned} \tag{2.1}$$

The existence of a unique solution is justified by Lemma 2.1. Moreover, we introduce the operator S^* as the solution operator of the adjoint equation, i.e., $p = S^*(y - y_d)$. Note that the operator S^* is the adjoint operator of S, if the operator S is investigated as an operator acting in $L^2(\Omega)$.

Due to Lemma 2.1, the adjoint equation admits a unique solution in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$, if $y_d \in L^p(\Omega)$. This space is embedded in $C^{0,1}(\bar{\Omega})$ for $p > N$.

In the sequel, we will use the following notation. The optimal control is denoted by $\bar{u}$. The optimal state $\bar{y} := S\bar{u}$ denotes the corresponding solution of (1.2) and the adjoint state $\bar{p} := S(\bar{y} - y_d)$ means the corresponding solution of (2.1).

Introducing the projection

$$\Pi_{[a,b]}(f(x)) = \max(a, \min(b, f(x))),$$

we can formulate the necessary and sufficient first-order optimality condition for (1.1)–(1.3).

LEMMA 2.2. *The variational inequality*

$$(\bar{p} + \nu \bar{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \textit{for all } u \in U_{\text{ad}} \tag{2.2}$$

is necessary and sufficient for the optimality of $\bar{u}$. This condition can be expressed equivalently by

$$\bar{u} = \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}\right). \tag{2.3}$$

Proof. Since the optimal control problem is strictly convex and radially bounded, we obtain the existence of a unique optimal solution. The optimality condition can be formulated as variational inequality (2.2). A standard pointwise a.e. discussion of this variational inequality leads to the projection formula (2.3), see [12]. ◻

We are now able to introduce the discretized problem. We define a finite-element based approximation of the optimal control (1.1)–(1.3). To this aim, we consider a family of triangulations $(T_h)_{h>0}$ of $\bar{\Omega}$. With each element $T \in T_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. The mesh size of the grid is defined by $h = \max_{T \in T_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(A1) There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma, \qquad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in T_h$ and all $h > 0$.

(A2) Let us define $\bar{\Omega}_h = \bigcup_{T \in T_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and that the vertices of T_h placed on the boundary of Γ_h are points of Γ. From [14], estimate (5.2.19), it is known that

$$|\Omega \setminus \Omega_h| \leq Ch^2,$$

where $|.|$ denotes the measure of the set. Moreover, we set

$$\begin{aligned} U_h &= \{v_h \in L^\infty(\Omega) :\ v_h \in \mathcal{P}_1 \text{ for all } T \in T_h, \}, \quad U_h^{ad} = U_h \cap U_{ad}, \\ V_h &= \{v_h \in C(\bar{\Omega}) :\ v_h \in \mathcal{P}_1 \text{ for all } T \in T_h, \text{ and } v_h = 0 \text{ on } \bar{\Omega} \setminus \Omega_h\}, \end{aligned}$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. For each $u_h \in U_h$, we denote by y_h the unique element of V_h that satisfies

$$a(y_h, v_h) = \int_\Omega u_h v_h \, dx \qquad \forall v_h \in V_h, \tag{2.4}$$

where $a :\ V_h \times V_h \to \mathbb{R}$ is the bilinear form defined by

$$a(y_h, v_h) = \int_\Omega \left(a_0(x) y_h(x) v_h(x) + \sum_{i,j=1}^{2} a_{ij}(x) D_i y_h(x) D_j v_h(x) \right) \, dx.$$

In other words, y_h is the approximated state associated with u_h. Via this equation, we define a solution operator S_h, $y_h = S_h u_h$. Moreover, because of $y_h = v_h = 0$ on $\bar{\Omega} \setminus \Omega_h$ the integrals over Ω can be replaced by integrals over Ω_h. The finite dimensional approximation of the optimal control problem is defined by

$$\inf J_h(u_h) = \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)} \qquad u_h \in U_h^{ad}. \tag{2.5}$$

The adjoint equation is discretized in the same way

$$a(p_h, v_h) = \int_\Omega (y_h - y_d) v_h \, dx \qquad \forall v_h \in V_h. \tag{2.6}$$

We define the operator S_h^* by the relation $p_h = S_h^*(y_h - y_d)$. Note, that the operators S_h and S_h^* can also be investigated as operators acting in $L^2(\Omega)$.

For our superconvergence result we need an additional assumption for $\bar{u}$. We know already that the associated adjoint state $\bar{p}$ belongs to a space $W^{2,p}(\Omega)$ for a certain $p > 2$. The optimal control $\bar{u}$ is obtained by the projection formula (2.3). Therefore, the optimal control $\bar{u}$ is a continuous function and we can differ between inactive point (i.e. $\bar{u}(x) \in (a, b)$) and active points $\bar{u}(x) \in \{a, b\}$. Hence, we can classify the triangles T_i in two sets K_1 and K_2:

$$\begin{aligned} K_1 &= \{T_i :\ T_i \text{ contains active } \textbf{and} \text{ inactive points}\}, \\ K_2 &= \{T_i :\ T_i \text{ contains only active or only inactive points}\}. \end{aligned} \tag{2.7}$$

The set K_2 covers the smooth part of $\bar{u}$, i.e. the optimal control belongs to the space $W^{2,p}(K_2)$. In contrast to this, the set K_1 contains the Lipschitz-part of $\bar{u}$, since $W^{2,p}(\Omega)$ is embedded in $C^{0,1}(\bar{\Omega})$ and the projection operator is continuous from $C^{0,1}(\bar{\Omega})$ to $C^{0,1}(\bar{\Omega})$. Clearly, the number of triangles in K_1 grows for decreasing h. Nevertheless, the following additional assumption is fulfilled in many practical cases:

(A3) $|K_1| \leq c \cdot h$.

Let $\bar{u}$ be the optimal solution of (1.1)–(1.3) with associated state $\bar{y} = S\bar{u}$ and adjoint state $\bar{p} = S^*(\bar{y} - y_d)$. Next, we denote the optimal solution of (2.5) by $\bar{u}_h$. Morover, we introduce the associated discretized state $\bar{y}_h = S_h\bar{u}_h$ and the corresponding discretized adjoint state $\bar{p}_h = S_h^*(\bar{y}_h - y_d)$.

Similar to [13], we propose a post-processing step. We start by the optimal solution $\bar{u}_h$. Although this control has only approximation rate $h^{3/2}$ (see [16]), we will prove that the approximation rate of the state and the adjoint state is even h^2:

LEMMA 2.3. *Assume that the assumptions (A1)–(A3) hold. Then the estimates*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2 \tag{2.8}$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le ch^2 \tag{2.9}$$

are valid.

The control $\tilde{u}_h$ is calculated by a projection of the discrete adjoint state $p_h(u_h)$ to the admissible set

$$\tilde{u}_h(x) = \Pi_{[a,b]}(-\frac{1}{\nu}(\bar{p}_h)(x)).$$

Now, we are able to state the main result.

THEOREM 2.4. *Let $\tilde{u}$ be the control constructed above. Under the assumptions (A1)–(A3) we obtain the error estimate*

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \le ch^2. \tag{2.10}$$

The proofs of Lemma 2.3 and Theorem 2.4 are derived in Section 4.

3. Finite element estimates.

In this section, we collect results from the finite element theory for elliptic equations.

LEMMA 3.1. *The norms of the discrete solution operators S_h and S_h^* are bounded,*

$$\begin{aligned} \|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c,\\ \|S_h\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} &\le c,\\ \|S_h\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c,\\ \|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, \end{aligned}$$

where c is, as always, independent of h.

The proof of this standard result can be found in several books about finite elements, for instance [3],[6].

LEMMA 3.2. *Let $f \in L^2(\Omega)$ be any function. The discretization error can be estimated by*

$$\|Sf - S_hf\|_{L^2(\Omega)} \le ch^2\|f\|_{L^2(\Omega)}, \tag{3.1}$$

$$\|S^*f - S_h^*f\|_{L^2(\Omega)} \le ch^2\|f\|_{L^2(\Omega)}. \tag{3.2}$$

For the proof we refer to [3],[6].

An estimate in the L^∞-norm plays an important part in the proof of the main result. Hence we recall a corresponding result from [16].

Lemma 3.3. *[16] Let $\bar{u}$ and $\bar{u}_h$ be the optimal solutions of the undiscretized problem and the discretized problem, respectively. Then an estimate*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega)} \leq c_\infty h \tag{3.3}$$

holds true with a positive constant c_∞.

Next, we will introduce an auxiliary function $w_h \in U_h^{ad}$ by

$$w_h(x) = \begin{cases} \bar{u}_h(x) & \text{if } x \in T_i \subset K_1 \\ (i_h\bar{u})(x) & \text{if } x \in T_i \subset K_2 \end{cases} \tag{3.4}$$

with the set K_1 and K_2 introduced in (2.7). Here, $i_h\bar{u}$ denotes the linear interpolate of the control $\bar{u}$ on the triangle T_i.

Let us comment this choice of the auxiliary function w_h. This function approximates the continuous optimal control in order h^2 in the L^2-norm on the smooth parts of the continuous control. The approximation order is h on the critical set K_1 (containing the kinks). This is used in the next lemma. Moreover, on the set K_1 the auxiliary function coincides with the discretized solutions. This property is the key point in the proof of the main results, since we can drop corresponding terms in certain variational inequalities.

Moreover, the auxiliary function w_h is defined as a discontinuous function. Therefore, the following proving technique cannot be applied for piecewise linear and continuous controls.

Lemma 3.4. *The estimate*

$$(v_h, \bar{u} - w_h)_{L^2(\Omega)} \leq ch^2\|v_h\|_{L^\infty(\Omega)} \left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + c_\infty\right) \tag{3.5}$$

is valid for all $v_h \in V_h$, provided that Assumptions (A1)–(A3) are fulfilled.

Proof. With the sets K_1 and K_2 introduced in (2.7) we have

$$(v_h, \bar{u} - w_h)_{L^2(\Omega)} = \int_{K_1} v_h(\bar{u} - w_h)\, dx + \int_{K_2} v_h(\bar{u} - w_h)\, dx. \tag{3.6}$$

The K_1-part can be estimated by

$$\left|\int_{K_1} v_h(\bar{u} - w_h)\, dx\right| \leq \|v_h\|_{L^\infty(\Omega)}\|\bar{u} - \bar{u}_h\|_{L^\infty(K_1)}|K_1|. \tag{3.7}$$

using definition (3.4). Now, (3.3), and (A3) imply

$$\left|\int_{K_1} v_h(\bar{u} - w_h)\, dx\right| \leq cc_\infty h^2\|v_h\|_{L^\infty(\Omega)}. \tag{3.8}$$

For a triangle T of the K_2-part we have

$$\int_{K_2} v_h(\bar{u} - w_h)\, dx = \int_{K_2} v_h(\bar{u} - i_h\bar{u})\, dx$$

again using definition (3.4). Consequently we find

$$\int_{K_2} v_h(\bar{u} - w_h)\, dx \leq \|v_h\|_{L^2(\Omega)}\|\bar{u} - i_h\bar{u}\|_{L^2(K_2)}. \tag{3.9}$$

Each triangle T_i of the set K_2 contains only active or inactive points. In the active triangles we have $\bar{u} = a$ or $\bar{u} = b$. Consequently the expression $\bar{u} - i_h\bar{u}$ vanishes on such triangles. On the inactive triangles we can replace $\bar{u} - i_h\bar{u}$ by $-\frac{1}{\nu}(\bar{p} - i_h\bar{p})$. Together with Lemma 2.1, this implies

$$\|\bar{u} - i_h\bar{u}\|_{L^2(K_2)} \leq \frac{1}{\nu}\|\bar{p} - i_h\bar{p}\|_{L^2(K_2)} \leq ch^2\|\bar{p}\|_{H^2(\Omega)} \leq ch^2(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{3.10}$$

Combining (3.8),(3.9), and (3.10), the assertion is obtained. □

LEMMA 3.5. *Let w_h be the functions defined by* (3.4)*. In addition, we assume that the assumptions (A1)–(A3) are satisfied. Then the estimate*

$$\|S_h\bar{u} - S_hw_h\|_{L^2(\Omega)} \leq ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}) \tag{3.11}$$

holds true.

Proof. We start with

$$\begin{aligned}\|S_h\bar{u} - S_hw_h\|^2_{L^2(\Omega)} &= (S_h\bar{u} - S_hw_h, S_h\bar{u} - S_hw_h)_{L^2(\Omega)} \\ &= (S_h^*S_h\bar{u} - S_h^*S_hw_h, \bar{u} - w_h)_{L^2(\Omega)}\end{aligned} \tag{3.12}$$

Next, we use estimate (3.5) and obtain

$$\|S_h\bar{u} - S_hw_h\|^2_{L^2(\Omega)} \leq ch^2\|S_h^*S_h\bar{u} - S_h^*S_hw_h\|_{L^\infty(\Omega)}\left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + c_\infty\right).$$

Applying Lemma 3.1, we obtain

$$\|S_h^*S_h\bar{u} - S_h^*S_hw_h\|_{L^\infty(\Omega)} \leq c\|S_h\bar{u} - S_hw_h\|_{L^2(\Omega)}.$$

Combining the last two inequalities, we end up with

$$\|S_h\bar{u} - S_hw_h\|_{L^2(\Omega)} \leq ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)})$$

which is exactly the assertion. □

COROLLARY 3.6. *If the assumptions of Lemma 3.5 hold, then, we have*

$$\|S_h^*(S_h\bar{u} - y_d) - S_h^*(S_hw_h - y_d)\|_{L^2(\Omega)} \leq ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{3.13}$$

By means of Lemma 3.2, we obtain in addition

$$\|\bar{p} - S_h^*(S_hw_h - y_d)\|_{L^2(\Omega)} \leq ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{3.14}$$

4. Superconvergence properties.

In this section we will prove the main results.

LEMMA 4.1. *The inequality*

$$(\bar{p} + \nu\bar{u}, \bar{u}_h - w_h)_{L^2(\Omega)} \geq 0 \tag{4.1}$$

is satisfied.

Proof. By definition, we have $w_h = \bar{u}_h$ on the set K_1. Consequently this part of the inner product vanishes. The set K_2 contains two types of triangles. It holds $\bar{p} + \nu\bar{u} = 0$ on all triangles where no constraint is active. Consequently this part of the scalar product vanishes, too. It remains to discuss the triangles where a constraint is active. Here we have $\bar{u} = w_h = a$ or $\bar{u} = w_h = b$. In this case the optimality condition (2.2) for $\bar{u}$ implies the desired inequality. □

LEMMA 4.2. *Assume that the assumptions (A1)-(A3) are fulfilled. Then, we have*

$$\|w_h - \bar{u}_h\|_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{4.2}$$

Proof. We start with the optimality condition for $\bar{u}_h$

$$(\bar{p}_h + \nu\bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \ge 0 \qquad \text{for all } u_h \in U_h^{ad}. \tag{4.3}$$

This inequality is tested with w_h

$$(\bar{p}_h + \nu\bar{u}_h, w_h - \bar{u}_h)_{L^2(\Omega)} \ge 0. \tag{4.4}$$

Next, we add the inequalities (4.1) and (4.4) and obtain

$$(\bar{p} - \bar{p}_h + \nu(\bar{u} - \bar{u}_h), \bar{u}_h - w_h)_{L^2(\Omega)} \ge 0. \tag{4.5}$$

or equivalently

$$\nu(\bar{u} - \bar{u}_h, \bar{u}_h - w_h)_{L^2(\Omega)} + (\bar{p} - \bar{p}_h, \bar{u}_h - w_h)_{L^2(\Omega)} \ge 0. \tag{4.6}$$

Using the definition of w_h, we find

$$\begin{aligned} \nu(\bar{u} - \bar{u}_h, \bar{u}_h - w_h)_{L^2(\Omega)} &= -\nu\|\bar{u}_h - w_h\|^2_{L^2(\Omega)} + \nu(\bar{u} - w_h, \bar{u}_h - w_h)_{L^2(\Omega)} \\ &= -\nu\|\bar{u}_h - w_h\|^2_{L^2(\Omega)} + \nu(\bar{u} - i_h\bar{u}, \bar{u}_h - w_h)_{L^2(K_2)} \end{aligned} \tag{4.7}$$

since the K_1-part vanishes because of $\bar{u}_h = w_h$ on K_1. The set K_2 contains the smooth parts of the control. Consequently, we find

$$\nu(\bar{u} - i_h\bar{u}, \bar{u}_h - w_h)_{L^2(K_2)} \le ch^2\|\bar{u}\|_{H^2(\Omega)}\|\bar{u}_h - w_h\|_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{H^2(\Omega)}\|\bar{u}_h - w_h\|_{L^2(\Omega)}.$$

Inserting this formula and (4.7) in (4.6), we get

$$\nu\|\bar{u}_h - w_h\|^2_{L^2(\Omega)} - (\bar{p} - \bar{p}_h, \bar{u}_h - w_h)_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{H^2(\Omega)}\|\bar{u}_h - w_h\|_{L^2(\Omega)} \tag{4.8}$$

Next, we estimate the inner product in (4.8)

$$\begin{aligned} (\bar{p} - \bar{p}_h, \bar{u}_h - w_h)_{L^2(\Omega)} &= (\bar{p} - S_h^*(S_h\bar{u}_h - y_d), \bar{u}_h - w_h)_{L^2(\Omega)} \\ &= (\bar{p} - S_h^*(S_h w_h - y_d), \bar{u}_h - w_h)_{L^2(\Omega)} \qquad (4.9) \\ &\quad + (S_h^*(S_h w_h - y_d) - S_h^*(S_h\bar{u}_h - y_d), \bar{u}_h - w_h)_{L^2(\Omega)}. \qquad (4.10) \end{aligned}$$

Now, Corollary 3.6 delivers the estimate for (4.9)

$$(\bar{p} - S_h^*(S_h w_h - y_d), w_h - \bar{u}_h)_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)})\|\bar{u}_h - w_h\|_{L^2(\Omega)}. \tag{4.11}$$

The second term (4.10) is estimated as follows

$$\begin{aligned} (S_h^*(S_h w_h - y_d) - S_h^*(S_h\bar{u}_h - y_d), \bar{u}_h - w_h)_{L^2(\Omega)} &= (S_h^* S_h(w_h - \bar{u}_h), \bar{u}_h - w_h)_{L^2(\Omega)} \\ &= (S_h(w_h - \bar{u}_h), S_h(\bar{u}_h - w_h))_{L^2(\Omega)} \\ &\le 0 \end{aligned} \tag{4.12}$$

Inserting (4.9)–(4.12) in (4.8), we obtain

$$\nu\|\bar{u}_h - w_h\|^2_{L^2(\Omega)} \le ch^2(\|\bar{p}\|_{H^2(\Omega)}\| + c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)})\|\bar{u}_h - w_h\|_{L^2(\Omega)}. \tag{4.13}$$

From Lemma 2.1 we get

$$\|\bar{p}\|_{H^2(\Omega)} \le c(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{4.14}$$

The last two inequalities imply the assertion. □

Now, we are ready to prove Lemma 2.3.

Proof. (Lemma 2.3) We start with

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le \|\bar{y} - S_h\bar{u}\|_{L^2(\Omega)} + \|S_h\bar{u} - S_h w_h\|_{L^2(\Omega)} + \|S_h w_h - \bar{y}_h\|_{L^2(\Omega)}. \tag{4.15}$$

The first term is estimated by Lemma 3.2. The second term was already investigated in Lemma 3.5. For the last term we use Lemma 3.1. Hence, we find

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}) + c\|w_h - \bar{u}_h\|_{L^2(\Omega)}. \tag{4.16}$$

Now, Lemma 4.2 implies

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{4.17}$$

It remains to show (2.9). Here, we find

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le \|\bar{p} - S_h^*(\bar{y} - y_d)\|_{L^2(\Omega)} + \|S_h^*(\bar{y} - y_d) - \bar{p}_h\|_{L^2(\Omega)}. \tag{4.18}$$

The first norm can be estimated by Lemma 3.2. For the second one we use (4.17) and Lemma 3.1. Consequently, we end up with

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}) \tag{4.19}$$

and the assertion is shown. □

We have already shown the superconvergence properties for the state and the adjoint state. This superconvergence is transferred to the control via the postprocessing.

Proof. (Theorem 2.4) By the definition of $\tilde{u}_h$, we obtain

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} = \left\|\Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}\right)\bar{u} - \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{p}_h\right)\right\|_{L^2(\Omega)}. \tag{4.20}$$

The projection operator is lipschitz continuous with constant 1. Consequently, we get

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \le \frac{1}{\nu}\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}. \tag{4.21}$$

Inserting (4.19), we end up with

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \le ch^2(c_\infty + \|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{4.22}$$

□

5. Numerical tests. Our approximation theory is tested for two examples where the exact solution of the undiscretized optimal control problem is known. These examples were originally introduced in [13].

In both cases, the Laplace operator $-\Delta$ was chosen for the elliptic operator A. The domain Ω is the unit square $(0,1)\times(0,1)$. We used uniform meshs, where the parameter N_h denotes the number of intervals in which the edges are divided. Hence, the quantities N_h and h are connected by the formula $N_h \cdot h = \sqrt{2}$. Both optimization problems were solved numerically by a primal-dual active set strategy, see [2] and [11]. The discretization was already described in Section 2: The state y and the adjoint state p were approximated by piecewise linear functions, whereas the control u is discretized by piecewise linear, but discontinuous functions. For comparison we also used piecewise constant functions for the control u.

The first example is a homogeneous Dirichlet problem, which fulfills the assumptions mentioned at the beginning of section 2, except the boundary regularity. Although Γ is not of class $C^{1,1}$, the $W^{2,p}$-regularity of $\bar{p}$ (see Lemma 2.1) is obtained by a result of Grisvard [9] for convex polygonal domain. In the second example, a Neumann boundary problem is studied. In this case, the theoretical results does not exactly fit to the problem. However, in the case $\Omega_h = \Omega$, the theory can be easily adapted.

Example 1. In this example, the Laplace equation with homogeneous Dirichlet boundary conditions is investigated, i.e. $a_0 \equiv 0$ in (1.2). Thus, the state equation is given by

$$\begin{aligned} -\Delta y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma. \end{aligned} \tag{5.1}$$

We define the optimal state by

$$\bar{y} = y_a - y_g$$

with an analytical part $y_a = \sin(\pi x_1)\sin(\pi x_2)$ and a less smooth function y_g, which is defined as the solution of

$$\begin{aligned} -\Delta y_g &= g && \text{in } \Omega \\ y_g &= 0 && \text{on } \Gamma. \end{aligned}$$

The function g is given by

$$g(x_1,x_2) = \begin{cases} u_f(x_1,x_2) - a & , \text{if } u_f(x_1,x_2) < a \\ 0 & , \text{if } u_f(x_1,x_2) \in [a,b] \\ u_f(x_1,x_2) - b & , \text{if } u_f(x_1,x_2) > b \end{cases}$$

with $u_f(x_1,x_2) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$. Due to the state equation (5.1), we obtain for the exact optimal control $\bar{u}$

$$\bar{u}(x_1,x_2) = \begin{cases} a & , \text{if } u_f(x_1,x_2) < a \\ u_f(x_1,x_2) & , \text{if } u_f(x_1,x_2) \in [a,b] \\ b & , \text{if } u_f(x_1,x_2) > b \end{cases} .$$

For the optimal adjoint state $\bar{p}$, we find

$$\bar{p}(x_1,x_2) = -2\pi^2\nu \sin(\pi x_1)\sin(\pi x_2).$$

Due to the adjoint state equation, we finally get

$$y_d(x_1,x_2) = \bar{y} + \Delta\bar{p} = y_a - y_g + 4\pi^4\nu \sin(\pi x_1)\sin(\pi x_2).$$

It can be easily shown, that these functions fulfill the necessary and sufficient first order optimality conditions. In the numerical tests, we chose $a = 3$, $b = 15$ and $\nu = 1$.

Figure 5.1 shows the approximation behavior of $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$. In the figures, $\bar{u}$ is denoted by u_{opt}.

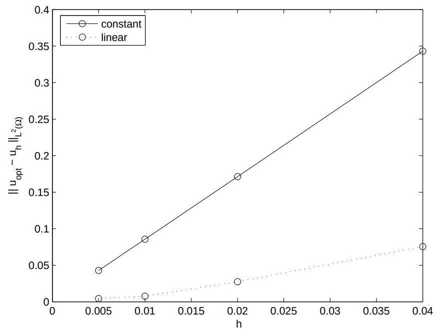

Fig. 5.1. $||\bar{u} - \bar{u}_h||_{L^2(\Omega)}$

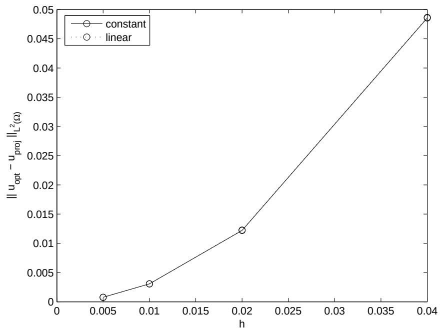

Fig. 5.2. $||\bar{u} - \tilde{u}||_{L^2(\Omega)}$

As mentioned in [16], the approximation in the L^2-norm is of order $h^{3/2}$ for the piecewise linear functions. In contrast to this, the approximation is only of order h for piecewise constant functions, see also Table 5.1. Figure 5.2 shows the convergence order after the projection. As one can see, the theoretical predictions are fulfilled and one obtains a quadratic approximation rate for $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$.

Table 5.1

	piecewise linear (discontinuous)			piecewise constant		
N_h	d.o.f.	$\|\|\bar{u} - \bar{u}_h\|\|_{L^2}$	$\|\|\bar{u} - \tilde{u}\|\|_{L^2}$	d.o.f.	$\|\|\bar{u} - \bar{u}_h\|\|_{L^2}$	$\|\|\bar{u} - \tilde{u}\|\|_{L^2}$
25	3456	0.07552	0.04866	1058	0.34312	0.04856
50	14406	0.02747	0.01227	4608	0.17124	0.01221
100	58806	0.00758	0.00306	19208	0.08563	0.00306
200	237606	0.00428	0.00076	78408	0.04288	0.00077

Example 2. We consider here the problem

$$\begin{aligned} -\Delta y + cy &= u \qquad \text{in } \Omega \\ \partial_n y &= 0 \qquad \text{on } \Gamma \end{aligned} \tag{5.2}$$

where ∂_n denotes the normal derivative with respect to the outward normal vector.

The optimal state $\bar{y} = y_a - y_g$ is constructed with $y_a(x_1, x_2) = \cos(\pi x_1)\cos(\pi x_2)$. The function y_g is determined by the equation

$$\begin{aligned} -\Delta y_g + cy_g &= g \qquad \text{in } \Omega \\ \partial_n y_g &= 0 \qquad \text{on } \Gamma, \end{aligned}$$

with the inhomogenity

$$g(x_1, x_2) = \begin{cases} u_f(x_1, x_2) - a & , \text{if } u_f(x_1, x_2) < a \\ 0 & , \text{if } u_f(x_1, x_2) \in [a, b] \\ u_f(x_1, x_2) & , \text{if } u_f(x_1, x_2) > b \end{cases}$$

and $u_f(x_1, x_2) = (2\pi^2 + c)\cos(\pi x_1)\cos(\pi x_2)$. The optimal control $\bar{u}$ is given by the state equation (5.2)

$$\bar{u}(x_1, x_2) = \begin{cases} a & , \text{ if } u_f(x_1, x_2) < a \\ u_f(x_1, x_2) & , \text{ if } u_f(x_1, x_2) \in [a, b] \\ b & , \text{ if } u_f(x_1, x_2) > b. \end{cases}$$

The optimal adjoint state is defined by

$$\bar{p}(x_1, x_2) = -(2\pi^2 + c)\nu \sin(\pi x_1)\sin(\pi x_2).$$

Moreover, the desired state y_d is chosen as

$$\begin{aligned} y_d(x_1, x_2) &= \bar{y} + \Delta\bar{p} - c\bar{p} \\ &= y_a - y_g + (4\pi^4\nu + 4\pi^2\nu c + \nu c^2)\sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

Again, it is easy to verify that these functions fulfill the necessary and sufficient first-order optimality conditions. In the numerical tests, we chose $a = -3$, $b = 15$ and $\nu = c = 1$.

Figure 5.3 and Figure 5.4 illustrate that the approximation behavior in the example with Neumann boundary conditions is similar to the example with Dirichlet boundary conditions.

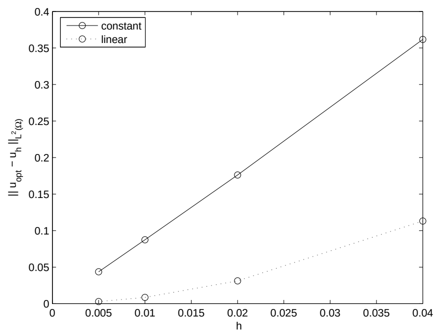

FIG. 5.3. $||\bar{u} - \bar{u}_h||_{L^2(\Omega)}$

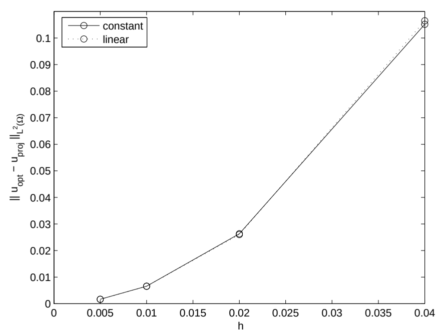

FIG. 5.4. $||\bar{u} - \tilde{u}||_{L^2(\Omega)}$

As one can see in Table 5.2, the absolute error is only slightly reduced by the projection.

TABLE 5.2

	piecewise linear (discontinuous)			piecewise constant		
N_h	d.o.f.	$\|\|\bar{u} - \bar{u}_h\|\|_{L^2}$	$\|\|\bar{u} - \tilde{u}\|\|_{L^2}$	d.o.f.	$\|\|\bar{u} - \bar{u}_h\|\|_{L^2}$	$\|\|\bar{u} - \tilde{u}\|\|_{L^2}$
25	3750	0.11312	0.10651	1250	0.36168	0.10517
50	15000	0.03121	0.02603	5000	0.17610	0.02632
100	60000	0.00856	0.00655	20000	0.08744	0.00656
200	240000	0.00283	0.00163	80000	0.04366	0.00164

Let us summarize our numerical experiences. The numerical experiments show the expected approximation rates. However, there are also surprising effects: Although the approximation behavior for piecewise linear and discontinuous controls is essentially better than for piecewise constant controls, the accuracy after the postprocessing is nearly the same. Therefore, the usage of piecewise constant controls seems to be more reasonable since the number of unknowns is smaller as for piecewise linear and discontinuous controls.

REFERENCES

[1] N. Arada, E. Casas, and F. Tröltzsch, *Error estimates for a semilinear elliptic optimal control problem*, Computional Optimization and Approximation, 23 (2002), pp. 201–229.

[2] M. Bergounioux, K. Ito, and K. Kunisch, *Primal-dual strategy for constrained optimal control problems*, SIAM Journal on Control and Optimization, 37 (1999), pp. 1176–1194.

[3] D. Braess, *Finite Elemente*, Springer-Verlag, Berlin Heidelberg, 1992.

[4] E. Casas, M. Mateos, and F. Tröltzsch, *Error estimates for the numerical approximation of boundary semilinear elliptic control problems*, Computational Optimization and Applications, 31 (2005), pp. 193–215.

[5] E. Casas and F. Tröltzsch, *Error estimates for linear-quadratic elliptic control problems*, in Analysis and Optimization of Differential Systems, Viorel Barbu, Irena Lasiecka, Dan Tiba and Constantin Varsan, ed., Boston, 2003, Kluwer Academic Publishers, pp. 89–100.

[6] P. Ciarlet, *Basic error estimates for elliptic problems*, in Finite Element Methods, vol. II of Handbook of Numerical Analysis, North-Holland, 1991, pp. 17–352.

[7] R. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, Journal of Mathematical Analysis and Applications, 44 (1973), pp. 28–47.

[8] T. Geveci, *On the approximation of the solution of an optimal control problem governed by an elliptic equation*, R.A.I.R.O. Analyse numeriqué, 13 (1979), pp. 313–328.

[9] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston-London-Melbourne, 1985.

[10] M. Hinze, *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Computational Optimization and Applications, 30 (2005), pp. 45–61.

[11] K. Kunisch and A. Rösch, *Primal-dual active set strategy for a general class of constrained optimal control problems*, SIAM Journal on Optimization, 13 (2002), pp. 321–334.

[12] K. Malanowski, *Convergence of approximations vs. regularity of solutions for convex, control–constrained optimal control problems*, Applied Mathematics and Optimization, 8 (1981), pp. 69–95.

[13] C. Meyer and A. Rösch, *Superconvergence properties of optimal control problems*, SIAM Journal on Control and Optimization, 43 (2004), pp. 970–985.

[14] P. Raviart and J. Thomas, *Introduction à l' Analyse Numérique des Équations aux Dérivées Partielles*, Masson, Paris, 1992.

[15] A. Rösch, *Error estimates for linear-quadratic control problems with control constraints*, Optimization Methods and Software, 21 (2006), pp. 121–134.

[16] A. Rösch and R. Simon, *Linear and discontinuous approximations for optimal control problems*, Numerical Functional Analysis and Optimization, 26 (2005), pp. 427–448.